ISRAEL JOURNAL OF MATHEMATICS 119 (2000), 349–377

DIVISORS ON VARIETIES OVER VALUATION DOMAINS

BY

HAGEN KNAF

Mathematisches Institut der Universität Heidelberg
Im Neuenheimer Feld 288, 69120 Heidelberg, Germany
e-mail: tengri@mathi.uni-heidelberg.de

ABSTRACT

Let $\mathcal{X}$ be a separated integral normal scheme of finite type over the valuation ring $\mathcal{O}_v$. It is shown that the set $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ of coherent fractionary $\mathcal{O}_{\mathcal{X}}$-ideals $\mathcal{J} \subseteq \underline{K(\mathcal{X})}$ satisfying the relation $\mathcal{J} = \widehat{\mathcal{J}} := (\mathcal{O}_{\mathcal{X}} : (\mathcal{O}_{\mathcal{X}} : \mathcal{J}))$ — the so-called divisorial $\mathcal{O}_{\mathcal{X}}$-ideals — forms a group with the composition law $(\mathcal{I}, \mathcal{J}) \mapsto \widehat{\mathcal{I} \cdot \mathcal{J}}$. This group posesses a natural embedding

$$\mathrm{div}: \mathcal{D}_{\mathrm{coh}}(\mathcal{X}) \to \mathrm{Div}(X) \oplus \prod_{\mathrm{v} \in \mathrm{V}} \mathrm{v}(K(\mathcal{X})),$$

where $\mathrm{Div}(X)$ denotes the group of Weil divisors of the generic fibre X of $\mathcal{X}|\,\mathrm{Spec}(\mathcal{O}_v)$, and V is a set of valuations of $K(\mathcal{X})$ determined by a subset of the generic points of the fibres $\mathcal{X} \times_{\mathcal{O}_v} \kappa(\mathcal{P})$, $\mathcal{P} \in \mathrm{Spec}(\mathcal{O}_v) \setminus 0$. The image $\mathrm{Div}(\mathcal{X})$ of div is proved to satisfy $\mathrm{Div}(\mathcal{X}) = \mathrm{Div}(X) \oplus \mathrm{Ver}(\mathcal{X})$ with a subgroup $\mathrm{Ver}(\mathcal{X}) \subseteq \prod_{\mathrm{v} \in \mathrm{V}} \mathrm{v}(K(\mathcal{X}))$. The structure of $\mathrm{Ver}(\mathcal{X})$ is determined provided that $\mathcal{X}$ satisfies additional conditions — for example, if $\mathcal{X}$ is projective over $\mathrm{Spec}(\mathcal{O}_v)$.

These facts are deduced from general results on the semigroup $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ of coherent divisorial $\mathcal{O}_{\mathcal{X}}$-ideals on an integral scheme $\mathcal{X}$: A criterion for $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ to be a group based on the notion of so-called Prüfer v-multiplication rings, and a valuation theoretic description of this group using valuations of $K(\mathcal{X})$ naturally associated to $\mathcal{X}$. The considerations leading to these results show that $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ can be understood as an ideal theoretic generalization of the group of Weil divisors on a normal noetherian scheme.

Following this idea a criterion for $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$, $\mathcal{X}$ a separated integral normal $\mathcal{O}_v$-scheme of finite type, to be equal to the group of Cartier divisors on $\mathcal{X}$ is given. The criterion is obtained by showing that for any point x on such a scheme the local generalized Weil divisor groups $\mathcal{D}_{\mathrm{coh}}(\mathrm{Spec}(\mathcal{O}_{\mathcal{X},x}))$ exist and by analyzing the structure of these groups.

Received March 24, 1999

Introduction

Let $\mathcal{O}_v$ be the valuation ring of a non-archimedian valuation v of a field K. Let $\mathcal{X}$ be an integral separated $\mathcal{O}_v$-scheme of finite type such that for some $n \in \mathbb{N}$ all irreducible components of all fibres of $\mathcal{X}|\operatorname{Spec}(\mathcal{O}_v)$ have dimension n. Such schemes will in the sequel be called $\mathcal{O}_v$-varieties.

The aim of this paper is to introduce a generalization of the group of Weil divisors for *normal* $\mathcal{O}_v$-varieties, and to study the structure of this group using valuations of the field F of rational functions on $\mathcal{X}$, that are naturally associated to $\mathcal{X}$.

The notion of Weil divisors can in fact be generalized to a much larger class of non-noetherian schemes than just $\mathcal{O}_v$-varieties: It is well-known that the Weil divisors of an integral separated normal noetherian scheme $\mathcal{X}$ are in bijection with the $\mathcal{O}_\mathcal{X}$-submodules $\mathcal{J}$ of the constant sheaf $\underline{F}$, that are of finite type and satisfy the relation

$$\mathcal{J} = \widehat{\mathcal{J}} := (\mathcal{O}_\mathcal{X} : (\mathcal{O}_\mathcal{X} : \mathcal{J})).$$

$\mathcal{O}_\mathcal{X}$-modules satisfying $\mathcal{J} = \widehat{\mathcal{J}}$ are called divisorial and the set of divisorial $\mathcal{O}_\mathcal{X}$-submodules of $\underline{F}$ is denoted by $\mathcal{D}_{\text{coh}}(\mathcal{X})$ — the meaning of the subscript will become clear some lines below.

$\mathcal{D}_{\text{coh}}(\mathcal{X})$ forms an abelian group with the composition $(\mathcal{I}, \mathcal{J}) \mapsto \widehat{\mathcal{I} \cdot \mathcal{J}}$. This group is isomorphic to the group of Weil divisors $\operatorname{Div}(\mathcal{X})$; the isomorphism is given using valuations: For any prime divisor P of $\mathcal{X}$ denote by v_P the corresponding discrete valuation of F. For any $\mathcal{J} \in \mathcal{D}_{\text{coh}}(\mathcal{X})$ the stalk $\mathcal{J}_x$ in the generic point $x \in P$ of P is a principal fractional $\mathcal{O}_{\mathcal{X},x}$-ideal $a_P\mathcal{O}_{\mathcal{X},x}$. One can therefore define $v_P(\mathcal{J}) := v_P(a_P)$. The map

$$\operatorname{div} : \mathcal{D}_{\text{coh}}(\mathcal{X}) \to \operatorname{Div}(\mathcal{X}),\ \mathcal{J} \mapsto (v_P(\mathcal{J}))_{P \text{ prime divisor of } \mathcal{X}}$$

is an isomorphism of groups — see [Fos], Ch. I.

The first section of the present article deals with the problem of adapting this approach to Weil divisors to non-noetherian schemes. More precisely let $\mathcal{X}$ be an integral scheme with coherent structure sheaf and field of rational functions F. The set $\mathcal{D}_{\text{coh}}(\mathcal{X})$ of coherent divisorial $\mathcal{O}_\mathcal{X}$-submodules $\mathcal{J} \subseteq \underline{F}$ is an abelian *semigroup* with the composition law given above. Coherence is needed in the definition to ensure that divisoriality is a local notion.

$\mathcal{D}_{\text{coh}}(\mathcal{X})$ is the candidate for the sheaf theoretic side of the generalization of Weil divisors. Weil divisors are required to form a group rather than a semigroup; the first of the two main results in section 1 deals with this aspect:

(1) *Let $\mathcal{X}$ be an integral scheme with coherent structure sheaf. Let* $\mathrm{P}(\mathcal{X})$ *be the set of points $x \in \mathcal{X}$ such that the maximal ideal $\mathcal{M}_{\mathcal{X},x}$ of the local ring $\mathcal{O}_{\mathcal{X},x}$ is an associated prime of the $\mathcal{O}_{\mathcal{X},x}$-module $F/\mathcal{O}_{\mathcal{X},x}$ — see subsection* 1.1 *for the definition.*

Assume that $\mathcal{X}$ satisfies the condition

$$(\star) \qquad \forall x \in \mathrm{P}(\mathcal{X})\colon \mathcal{O}_{\mathcal{X},x} \text{ is a valuation ring.}$$

Then $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ is a group.

The condition $(\star)$ is the non-noetherian analogue of being regular in codimension one, which ensures the existence of the group of Weil divisors in the noetherian case.

Assuming $(\star)$ the structure of the group $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ can be described to some extent using certain valuations associated with $\mathcal{X}$:

(2) *Let $\mathcal{X}$ be an integral quasi-compact separated scheme with coherent structure sheaf. Then the set* $\mathrm{Val}(\mathcal{X}) := \{x \in \mathcal{X} \mid \mathcal{O}_{\mathcal{X},x} \text{ is a valuation ring}\}$ *has maximal elements with respect to specialization.*

For each maximal element x of $\mathrm{Val}(\mathcal{X})$ *choose a valuation of F with valuation ring $\mathcal{O}_{\mathcal{X},x}$ and denote the set of valuations obtained in this way by $\mathcal{V}(\mathcal{X})$. Then the map*

$$\mathrm{div}\colon \mathcal{D}_{\mathrm{coh}}(\mathcal{X}) \to \prod_{v \in \mathcal{V}(\mathcal{X})} vF$$

can be defined as in the noetherian case and gives an injective group homomorphism.

The image of div is denoted by $\mathrm{Div}(\mathcal{X})$ and should be understood as the analogue of the usual definition of Weil divisors in the noetherian case. Unfortunately the map div is in general not surjective, which restricts its use to determine the structure of $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$, and shows that this structure can be rather complicated.

The results of section 1 are consequences of the theory of Prüfer v-multiplication rings; the relevant parts of this theory are summarized in subsection 1.1.

In section 2 the general results of section 1 are applied to the class of $\mathcal{O}_v$-varieties as defined at the beginning of this introduction: As a direct consequence of results of M. Nagata and G. Sabbagh it is first shown that any $\mathcal{O}_v$-variety $\mathcal{X}$ has coherent structure sheaf and satisfies condition $(\star)$ if one furthermore requires $\mathcal{X}$ to be normal. Hence for a normal $\mathcal{O}_v$-variety, $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ turns out to be a group.

Understanding the structure of this group is then the main goal of subsection 2.1. In view of the result (2) this amounts to determining the set $\mathrm{Val}(\mathcal{X})$ resp. the set of valuations $\mathcal{V}(\mathcal{X})$ corresponding to the maximal elements of $\mathrm{Val}(\mathcal{X})$ with respect to specialization. Having done this one finally has to calculate the image of the div-map introduced in (2).

The determination of $\mathrm{Val}(\mathcal{X})$ is carried out using an affine generalization of Zariski's Main Theorem due to C. Peskine; the result is quite intuitive:

(3) *A normal $\mathcal{O}_v$-variety $\mathcal{X}$ satisfies*

$$\mathrm{Val}(\mathcal{X}) = X^{(1)} \cup \bigcup_{\mathcal{P} \in \mathrm{Spec}(\mathcal{O}_v) \setminus 0} \mathrm{Gen}(\mathcal{X} \times_{\mathcal{O}_v} \kappa(\mathcal{P})),$$

where $X^{(1)}$ denotes the generic points of the Weil prime divisors on the generic fibre X of $\mathcal{X}$, and $\mathrm{Gen}(\mathcal{X} \times_{\mathcal{O}_v} \kappa(\mathcal{P}))$ is the set of generic points of the fibre of $\mathcal{X}|\mathcal{O}_v$ over $\mathcal{P}$.

At least in the case of a closed structure morphism $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_v)$ it is now simple to write down the set $\mathcal{V}(\mathcal{X})$ and consequently the div-homomorphism:

(4) *Let $\mathcal{X}$ be a normal $\mathcal{O}_v$-variety with closed structure morphism and* V *the set of valuations of its field of rational functions F corresponding to the local rings $\mathcal{O}_{\mathcal{X},x}$, $x \in \mathrm{Gen}(\mathcal{X} \times_{\mathcal{O}_v} \kappa(\mathcal{M}_v))$, $\mathcal{M}_v \lhd \mathcal{O}_v$ the maximal ideal; then the* div-*homomorphism looks like*

$$\mathrm{div}\colon \mathcal{D}_{\mathrm{coh}}(\mathcal{X}) \to \mathrm{Div}(X) \oplus \bigoplus_{\mathrm{v} \in \mathrm{V}} \mathrm{v}(F),$$

where $\mathrm{Div}(X)$ *is the group of (ordinary) Weil divisors of the generic fibre X of $\mathcal{X}$.*

The image of div can be calculated (to a certain extent) due to the following fact, interesting in itself: The ideal sheaf $\mathcal{J}_P$ of the closed subscheme $\mathbf{P} \subseteq \mathcal{X}$ obtained by taking the Zariski closure of a Weil prime divisor P of the generic fibre X is a coherent divisorial $\mathcal{O}_{\mathcal{X}}$-ideal. This fact can be used to embed $\mathrm{Div}(X)$ into $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$; using the Prüfer ring $\mathcal{O}_{\mathrm{V}} := \bigcap_{\mathrm{v} \in \mathrm{V}} \mathcal{O}_{\mathrm{v}}$ the structure of $\mathrm{Div}(\mathcal{X})$ can now be described as follows:

(5) *Let $\mathcal{X}$ be a normal $\mathcal{O}_v$-variety with closed structure morphism, function field F and generic fibre X. Then:*

1. *There exists a subgroup* $\mathrm{Ver}(\mathcal{X}) \subseteq \prod_{\mathrm{v} \in \mathrm{V}} \mathrm{v} F$ *such that* $\mathrm{Div}(\mathrm{Spec}(\mathcal{O}_{\mathrm{V}})) \subseteq \mathrm{Ver}(\mathcal{X})$ *and* $\mathrm{Div}(\mathcal{X}) = \mathrm{Div}(X) \oplus \mathrm{Ver}(\mathcal{X})$.
2. *If the generic points of the closed fibre of $\mathcal{X}$ possess an affine open neighborhood one has* $\mathrm{Ver}(\mathcal{X}) = \mathrm{Div}(\mathrm{Spec}(\mathcal{O}_{\mathrm{V}}))$.

There is also a version of this result without the assumption of a closed structure morphism — see subsection 2.1.

The group of Cartier divisors $\mathrm{CaDiv}(\mathcal{X})$ on an integral scheme $\mathcal{X}$ can always be embedded into the semigroup $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ viewing Cartier divisors as invertible subsheaves of the constant sheaf $\underline{F}$.

In the final subsection 2.2 the local Weil divisor group $\mathrm{Div}(\mathrm{Spec}(\mathcal{O}_{\mathcal{X},x}))$ in a point x of a normal $\mathcal{O}_v$-variety $\mathcal{X}$ is investigated: The group $\mathrm{Div}(\mathrm{Spec}(\mathcal{O}_{\mathcal{X},x}))$ splits in a way analogous to (5 1.) into a horizontal part defined by certain prime divisors of the generic fibre of $\mathcal{X}$ and a vertical part defined by a subset of V. Using this result a criterion for the equation $\mathrm{CaDiv}(\mathcal{X}) = \mathrm{Div}(\mathcal{X})$ to hold is given. Since the statement of both results is quite technical we omit a summary at this point and refer the reader directly to subsection 2.2.

The investigation presented in this article was motivated by the results on curves over valuation rings and their connection to constant reductions of algebraic function fields of transcendence degree 1 obtained by the members of a working group around Prof. P. Roquette at the universities of Heidelberg and Bordeaux — see [G] and the references given in that paper. It is a generalization of a part of the author's doctoral thesis [Kn1].

Roquette's paper [Roq1] and Kani's thesis [Kan1] had a particular influence on the present work: In [Roq1] a comparison is made between the Weil divisors of the generic fibre of a projective $\mathcal{O}_v$-variety $\mathcal{X}$ with the Weil divisors of the closed fibre $\mathcal{X} \times_{\mathcal{O}_v} \mathcal{O}_v/\mathcal{M}_v$, provided the fibres of $\mathcal{X}$ are smooth.

Kani develops a divisor theory for field extensions $L|K$ equipped with a set M of K-trivial valuations of L. He uses it to investigate $\mathcal{O}_v$-curves $C \times_K \mathcal{O}_v$, $v \in M$, where C is a smooth proper K-curve, with the aim to give a so-called non-standard proof of the Mordell–Weil theorem for the jacobian of a curve.

ACKNOWLEDGEMENT: Parts of this paper were written during a visit at the University of Stellenbosch/South Africa. I want to express my thanks for the financial support I received from the university during the visit. Furthermore, I owe thanks to Barry Green for several helpful discussions on the content of this paper and to Oliver Delzeith for many conversations about divisoriality.

1. Divisorial ideal sheaves and divisors on integral schemes

The purpose of this section is to introduce and study the abelian semigroup $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ of coherent divisorial $\mathcal{O}_{\mathcal{X}}$-ideals on an integral scheme $\mathcal{X}$ with coherent structure sheaf. The main emphasis is put on the problems of finding properties of the scheme $\mathcal{X}$ ensuring that $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ is a group and to describe this group using valuations of the function field $K(\mathcal{X})$ naturally associated to $\mathcal{X}$. The discussion of these problems shows the beautiful analogy as well as the main differences to the case of separated noetherian normal schemes, in which $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ is isomorphic to the group of Weil divisors on $\mathcal{X}$, the isomorphism given via the discrete valuations associated to prime divisors of $\mathcal{X}$.

The results of this section are based on the theory of Prüfer v-multiplication rings. To improve accessibility of the article the section starts with a summary of the relevant results on this type of ring.

1.1 A SUMMARY OF THE AFFINE CASE. Throughout the whole subsection let A be a domain with field of fractions F. We will frequently consider valuations v of the field F, which are always assumed to be non-archimedean. Their valuation ring is denoted by $\mathcal{O}_v$, its maximal ideal by $\mathcal{M}_v$. Usually in this article valuation rings will occur as localizations of certain domains. Given a valuation ring $\mathcal{O}$ the associated valuation $v\colon F \to \Gamma$, Γ a totally ordered abelian group, is unique up to order preserving isomorphisms of Γ. The considerations, statements etc. in this article are independent of the particular choice of v; therefore by abuse of language we will talk about *the valuation v corresponding to $\mathcal{O}$*.

Denote by $\mathcal{F}(A)$ the set of non-zero fractional ideals of A. In the sequel we will consider several subsets of $\mathcal{F}(A)$ that can be best defined in terms of $*$-operations on $\mathcal{F}(A)$: A **$*$-operation** on $\mathcal{F}(A)$ is a map $\mathcal{F}(A) \to \mathcal{F}(A), I \mapsto I^*$ with the properties

1. $\forall I \in \mathcal{F}(A)\colon I^{**} = I^*$,
2. $(\forall I \in \mathcal{F}(A)\colon I \subseteq I^*)$ and $(I \subseteq J^* \Rightarrow I^* \subseteq J^*)$,
3. $\forall x \in A,\ I \in \mathcal{F}(A)\colon (xA)^* = xA,\ \ (xI)^* = xI^*$.

Fractional ideals with the property $I = I^*$ are called **$*$-ideals**. A $*$-ideal I with the property $I = I_0^*$, I_0 a finitely generated fractional ideal, is called **$*$-finite**. The sets of $*$-ideals resp. $*$-finite ideals are denoted by $\mathcal{D}^*(A)$ resp. $\mathcal{D}^*_{\mathrm{fin}}(A)$. They both form abelian semigroups with the composition

$$(I, J) \mapsto (IJ)^*$$

and A is the neutral element. This composition is well defined due to the equations

$$(IJ)^* = (IJ^*)^* = (I^*J^*)^*.$$

See [Gil], Ch. V, §32 for the basic theory of $*$-operations.

One of the most important examples of $*$-operations is given by the map

$$\mathcal{F}(A) \to \mathcal{F}(A),\ \ I \mapsto \widehat{I} := (A : (A : I)).$$

It is usually called the **v-operation**. The corresponding semigroups are denoted by $\mathcal{D}^v(A)$ and $\mathcal{D}^v_{\mathrm{fin}}(A)$. To emphasize their connection with divisor theory we will follow those authors calling the elements of these semigroups **divisorial** resp. **divisorially finitely generated** ideals rather than v- and v-finite ideals. See [Fos], Ch. I or [Gil], Ch. V, §34 for the basics on divisorial ideals.

One of the tasks in multiplicative ideal theory is to determine those domains A for which $\mathcal{D}^v(A)$ resp. $\mathcal{D}^v_{\text{fin}}(A)$ form *groups*. It is well-known that $\mathcal{D}^v(A)$ is a group iff A is completely integrally closed, a property which is too restrictive for our purposes.

A characterization of those domains A for which $\mathcal{D}^v_{\text{fin}}(A)$ is a group is also known and the corresponding class of domains is more interesting. Following the literature such a domain is called a **Prüfer v-multiplication ring** (abbreviated: **PvM-ring**). To formulate the characterization of PvM-rings one needs (weakly) associated primes as introduced in [Bou], Ch. VI: Let M be an A-module. The prime ideal $\mathcal{P} \in \operatorname{Spec}(A)$ is called an **associated prime of M** if there exists $m \in M \setminus 0$ such that

$$\mathcal{P} \supseteq \operatorname{Ann}_A(m) = \{a \in A \mid am = 0\}$$

and $\mathcal{P}$ is minimal subject to this condition.

We will use associated primes of the A-module F/A. The set of associated primes of this module is denoted by $\mathrm{P}(A)$. Any $\mathcal{P} \in \mathrm{P}(A)$ is a minimal prime ideal over an ideal of the form $(aA : bA)$, $a, b \in A$, $b \notin aA$.

The relevance of associated primes of F/A for questions concerning divisoriality is shown by the following results:

1.1 ([Gla], Lemma 6.2.6, [Vas2], Prop. 3.3): *In any domain A one has:*

1. $\forall I \in \mathcal{D}^v(A)$: $I = \bigcap_{\mathcal{P} \in \mathrm{P}(A)} IA_{\mathcal{P}}$. *In particular this holds for A itself.*
2. $\forall I, J \in \mathcal{D}^v(A)$: $I = J \Leftrightarrow IA_{\mathcal{P}} = JA_{\mathcal{P}}\ \forall \mathcal{P} \in \mathrm{P}(A)$.

Remark: The two articles cited both assume A to be coherent, which is not necessary.

The desired characterization of PvM-rings is the content of the following result:

1.2 ([Mot-Zaf], Theorem 3.2 and [Zaf], Theorem 2): *Let A be a domain.*

1. *A is a PvM-ring iff the following conditions are satisfied:*
   (a) $\forall \mathcal{P} \in \mathrm{P}(A)$: *$A_{\mathcal{P}}$ is a valuation ring,*
   (b) $\forall a, b \in A$: $aA \cap bA \in \mathcal{D}^v_{\text{fin}}(A)$.
2. *If A is normal and $aA \cap bA$ is finitely generated for any pair $a, b \in A$ then A is a PvM-ring.*

Note that due to (1.1 1.) a consequence of (1.2 1 (a)) is that PvM-rings are normal.

A special case of (1.2 2.) is of particular interest in this article: Recall that an A-module M is called **coherent** if it is finitely generated and every finitely generated submodule of M is of finite presentation. A ring A is called a **coherent ring** if it is a coherent A-module, that is if every finitely generated ideal of A is finitely presented. In a coherent ring A the intersection of two finitely generated ideals is finitely generated ([Gla], Th. 2.3.2); hence one deduces from (1.2):

1.3: *Every normal coherent domain is a PvM-ring.*

This result includes the case of normal noetherian domains, where $\mathrm{P}(A)$ consists exactly of the prime ideals of height 1 and $\mathcal{D}^v_{\mathrm{fin}}(A)$ is the free abelian group over $\mathrm{P}(A)$. The latter is proved by using the discrete valuations corresponding to the local rings $A_{\mathcal{P}}$, $\mathcal{P} \in \mathrm{P}(A)$. A similiar procedure can be applied in the case of PvM-rings, but the situation is more complicated.

Let A be a normal domain and $\mathcal{V}$ a family of valuations of F such that $A = \bigcap_{v\in\mathcal{V}} \mathcal{O}_v$. According to [Gil], Th. 32.5 the assignment

$$I \mapsto \bigcap_{v\in\mathcal{V}} I\mathcal{O}_v =: I^{\mathcal{V}}$$

is a $*$-operation. The corresponding two semigroups are denoted by $\mathcal{D}^{\mathcal{V}}(A)$ and $\mathcal{D}^{\mathcal{V}}_{\mathrm{fin}}(A)$. This $*$-operation has the property $I\mathcal{O}_v = I^{\mathcal{V}}\mathcal{O}_v$ for every $I \in \mathcal{F}(A)$, $v \in \mathcal{V}$. In particular, for any $I \in \mathcal{D}^{\mathcal{V}}_{\mathrm{fin}}(A)$,

$$v(I) := \min(va|\, a \in I)$$

exists. This gives the opportunity to define a map

$$\text{(1)} \qquad \operatorname{div}: \mathcal{D}^{\mathcal{V}}_{\mathrm{fin}}(A) \to \prod_{v\in\mathcal{V}} vF, \quad I \mapsto (vI)_{v\in\mathcal{V}}.$$

div is an injective homomorphism of semigroups. In general div is not surjective — see [End], Ex. II-18 for counterexamples.

The link between the $*$-operation $I \mapsto I^{\mathcal{V}}$ and the v-operation is given by the following surprising result:

1.4 ([Gil], Prop. 44.13): *Let A be a normal domain and $\mathcal{V}$ a family of valuation rings such that $A = \bigcap_{v\in\mathcal{V}} \mathcal{O}_v$ and $\mathcal{O}_v = A_{A\cap\mathcal{M}_v}$ for all $v \in \mathcal{V}$. Then for any finitely generated fractional ideal I the equation $I^{\mathcal{V}} = \widehat{I}$ holds, i.e. $\mathcal{D}^{\mathcal{V}}_{\mathrm{fin}}(A) = \mathcal{D}^v_{\mathrm{fin}}(A)$ as semigroups.*

As an important consequence of this theorem formula (1) gives a description of the structure of $\mathcal{D}^v_{\mathrm{fin}}(A)$ in terms of valuations of the field of fractions of A. In the case A is a PvM-ring one could use the family $\mathcal{V}$ of valuations of F corresponding to the local rings $A_{\mathcal{P}}$, $\mathcal{P} \in \mathrm{P}(A)$, as is suggested by (1.2 1.) and

(1.1 1.). Since there may be containment relations among the primes in $\mathrm{P}(A)$ this decision would have the disadvantage of encorporating redundant information into the homomorphism div corresponding to the family $\mathcal{V}$. One therefore follows another track to find a suitable family $\mathcal{V}$:

The map

$$\mathcal{F}(A) \to \mathcal{F}(A),\ I \mapsto I^t := \bigcup_{I_0 \subseteq I \text{ finitely generated}} \widehat{I_0}$$

is a $*$-operation called the **t-operation** and defines the semigroup $\mathcal{D}^t(A)$ of t-ideals. The set of integral t-ideals $I \lhd A$ has maximal elements with respect to inclusion. Every integral t-ideal $I \lhd A$ is contained in a maximal t-ideal and maximal t-ideals are prime. The set of maximal t-ideals will be denoted by $\mathrm{tMax}(A)$.

Following M. Zafrullah we furthermore call a prime $\mathcal{P} \in \mathrm{Spec}(A)$ a **valued prime** if the local ring $A_{\mathcal{P}}$ is a valuation ring. The set of valued primes is denoted by $\mathrm{Val}(A)$ or $\mathrm{Val}(\mathrm{Spec}(A))$, if we want to emphasize the geometric point of view.

The use of t-ideals for our purpose is summarized in the following results of M. Griffin, J. L. Mott and M. Zafrullah:

1.5 ([Gri1], Prop. 4, [Gri1], Th. 5, [Mot-Zaf], Prop. 4.1): *Let A be a domain. Then:*

1. $A = \bigcap_{\mathcal{P} \in \mathrm{tMax}(A)} A_{\mathcal{P}}$,
2. *A is a PvM-ring iff* $\mathrm{tMax}(A) \subseteq \mathrm{Val}(A)$,
3. *A is a PvM-ring iff* $\mathrm{Val}(A) = \mathrm{Spec}(A) \cap \mathcal{D}^t(A)$.

From this result the following information relevant for this article can be extracted: For a PvM-ring A the set $\mathrm{Val}(A)$ has maximal elements and the set of maximal elements equals $\mathrm{tMax}(A)$ (1.5 2.+3.). Furthermore, by (1.5 1.) and (1.4) the set of valuations defined by the valuation rings $A_{\mathcal{P}}, \mathcal{P} \in \mathrm{tMax}(A)$, can be used to define a homomorphism div via formula (1) describing $\mathcal{D}^v_{\mathrm{fin}}(A)$. Note that there are no containment relations among the members of $\mathrm{tMax}(A)$. The special family of valuations defined by $\mathrm{tMax}(A)$ will in the sequel be denoted by $\mathcal{V}(A)$.

Beside the normal coherent domains the so-called rings of Krull type form another large class of PvM-rings. They will occur as local rings of certain schemes in this article. A **ring of Krull type** is a domain A such that there exists a family of valuations $\mathcal{V}$ of its field of fractions F with the properties

1. $A = \bigcap_{v \in \mathcal{V}} \mathcal{O}_v$,
2. for every $v \in \mathcal{V}$: $\mathcal{O}_v = A_{\mathcal{M}_v \cap A}$,

3. for every $a \in A \setminus 0$ the set $\{v \in \mathcal{V} | \, va > 0\}$ is finite.

The family $\mathcal{V}$ is called a **defining family for** A.

1.6 ([Gri1], Theorem 7): *Every ring of Krull type is a PvM-ring.*

Applying (1.4) to the defining family $\mathcal{V}$ of a ring of Krull type gives a homomorphism

$$\operatorname{div}: \mathcal{D}^v_{\text{fin}}(A) \to \bigoplus_{v \in \mathcal{V}} vF$$

into the direct *sum* instead of the direct product of the value groups. This is due to condition 3 in the definition and more similiar to the noetherian case. Note also that there are many defining families of one and the same ring of Krull type.

1.2 The case of integral schemes. Throughout the whole subsection let $\mathcal{X}$ be an integral scheme with field of rational functions F. We will furthermore need to assume that the structure sheaf $\mathcal{O}_\mathcal{X} \subset \underline{F}$ is a *coherent* sheaf; here $\underline{F}$ denotes the constant sheaf over F.

We will frequently use two facts about coherent $\mathcal{O}_\mathcal{X}$-submodules $\mathcal{J} \subset \underline{F}$:

(A) If the structure sheaf $\mathcal{O}_\mathcal{X}$ is coherent, then for every affine open $U \subseteq \mathcal{X}$ the ring $\mathcal{O}_\mathcal{X}(U)$ is a coherent ring. The affine scheme $\mathcal{X} = \operatorname{Spec}(A)$ has coherent structure sheaf iff the ring A is coherent.

(B) On an integral scheme $\mathcal{X}$ with coherent structure sheaf a $\mathcal{O}_\mathcal{X}$-submodule $\mathcal{J} \subseteq \underline{F}$ is coherent iff it is of finite type.

Property (B) is proved as follows: Since coherence is a local property it suffices to treat the affine case $\mathcal{X} = \operatorname{Spec}(A)$. In this case a $\mathcal{O}_\mathcal{X}$-submodule $\mathcal{J} \subseteq \underline{F}$ of finite type has the form $\widetilde{J}$, where J is a finitely generated A-submodule of F. It is well-known ([Gla], Th. 2.3.2 (3)) that torsion free finitely generated modules over a coherent ring are coherent, which proves the assertion.

$\mathcal{O}_\mathcal{X}$-modules $\mathcal{J} \subseteq \underline{F}$ are usually called **fractional $\mathcal{O}_\mathcal{X}$-ideals**. If $\mathcal{I}, \mathcal{J}$ are fractional $\mathcal{O}_\mathcal{X}$-ideals one can form the fractional $\mathcal{O}_\mathcal{X}$-ideals $\mathcal{I} \cdot \mathcal{J}$ and $(\mathcal{I} : \mathcal{J})$. We define

$$\widehat{\mathcal{J}} := (\mathcal{O}_\mathcal{X} : (\mathcal{O}_\mathcal{X} : \mathcal{J}))$$

for any fractional $\mathcal{O}_\mathcal{X}$-ideal $\mathcal{J}$ and call $\mathcal{J}$ a **divisorial $\mathcal{O}_\mathcal{X}$-ideal** if the equation

$$\mathcal{J} = \widehat{\mathcal{J}}$$

holds.

In this generality the concept of divisoriality does not behave nicely, since the operation $\mathcal{J} \mapsto \widehat{\mathcal{J}}$ does not commute with taking stalks. Therefore we restrict our considerations to *coherent* or, what by (B) amounts to the same, fractional $\mathcal{O}_\mathcal{X}$-ideals of finite type: Let $\mathcal{D}_{\text{coh}}(\mathcal{X})$ denote the set of coherent divisorial $\mathcal{O}_\mathcal{X}$-ideals. The elements of $\mathcal{D}_{\text{coh}}(\mathcal{X})$ *are* characerized by their local behavior:

LEMMA 1.7: *Let $\mathcal{J}$ be a coherent fractional $\mathcal{O}_\mathcal{X}$-ideal. Then:*

1. $\forall x \in \mathcal{X}$: $(\widehat{\mathcal{J}})_x = \widehat{\mathcal{J}_x}$,
2. $\mathcal{J} \in \mathcal{D}_{\text{coh}}(\mathcal{X}) \Leftrightarrow \forall x \in \mathcal{X}$: $\mathcal{J}_x \in \mathcal{D}^v_{\text{fin}}(\mathcal{O}_{\mathcal{X},x})$.

In particular, for any coherent fractional $\mathcal{O}_\mathcal{X}$-ideal $\mathcal{J}$ one has $\widehat{\mathcal{J}} \in \mathcal{D}_{\text{coh}}(\mathcal{X})$.

Proof: 1. It is straightforward to prove that for a $\mathcal{O}_\mathcal{X}$-ideal $\mathcal{J} \subseteq \underline{F}$ of finite type the equations $(\mathcal{O}_\mathcal{X} : \mathcal{J})_x = (\mathcal{O}_{\mathcal{X},x} : \mathcal{J}_x)$ for all $x \in \mathcal{X}$ hold. If in addition $\mathcal{J}$ is coherent then by [EGA1], 0, 5.3.8 $(\mathcal{O}_\mathcal{X} : \mathcal{J})$ is coherent too — in particular of finite type. This proves the assertion.

2. The implication $\Rightarrow$ follows from point 1. For any fractional $\mathcal{O}_\mathcal{X}$-ideal $\mathcal{J}$ the inclusion $\mathcal{J} \subseteq \widehat{\mathcal{J}}$ holds. Assuming divisoriality of all stalks $\mathcal{J}_x$ yields $\mathcal{J}_x = (\widehat{\mathcal{J}})_x$ — again due to point 1. ∎

By [EGA1], 0, 5.3.7+5.3.8 the fractional $\mathcal{O}_\mathcal{X}$-ideals $\mathcal{I}\mathcal{J}$ and $(\mathcal{I} : \mathcal{J})$ are coherent if $\mathcal{I}$ and $\mathcal{J}$ are coherent. This fact and Lemma 1.7 show that the map

$$\mathcal{D}_{\text{coh}}(\mathcal{X}) \times \mathcal{D}_{\text{coh}}(\mathcal{X}) \to \mathcal{D}_{\text{coh}}(\mathcal{X}), \quad (\mathcal{I}, \mathcal{J}) \mapsto \widehat{\mathcal{I}\mathcal{J}}$$

is welldefined. It gives $\mathcal{D}_{\text{coh}}(\mathcal{X})$ the structure of a commutative semigroup with neutral element $\mathcal{O}_\mathcal{X}$ — remember that $\mathcal{O}_\mathcal{X}$ is assumed to be coherent.

On an affine scheme $\mathcal{X} = \text{Spec}(A)$ with coherent structure sheaf a coherent fractional $\mathcal{O}_\mathcal{X}$-ideal $\mathcal{J}$ is divisorial iff it has the form $\mathcal{J} = \widetilde{J}$, where $J \subseteq F = \text{Quot}(A)$ is a coherent A-module satisfying $J = \widehat{J}$. Therefore one has:

1.8: *For the affine scheme $\mathcal{X} = \text{Spec}(A)$ the map*

$$\mathcal{D}_{\text{coh}}(\mathcal{X}) \to \mathcal{D}^v_{\text{fin}}(A)\ ,\ \mathcal{J} \mapsto \mathcal{J}(\mathcal{X})$$

is an isomorphism of semigroups. In particular the elements of $\mathcal{D}^v_{\text{fin}}(A)$ are finitely generated.

The semigroup $\mathcal{D}_{\text{coh}}(\mathcal{X})$ is the candidate for the sheaf theoretic side of our generalization of Weil divisors. These generalized Weil divisors should — as in the noetherian case — form a group, a requirement which forces us to generalize (1.2) to schemes:

Definition 1.9: A point x on an integral scheme $\mathcal{X}$ is called **associated point of $\mathcal{X}$** if $\mathcal{M}_{\mathcal{X},x} \in \mathrm{P}(\mathcal{O}_{\mathcal{X},x})$ holds for the maximal ideal $\mathcal{M}_{\mathcal{X},x}$ of the local ring $\mathcal{O}_{\mathcal{X},x}$. The set of associated points of $\mathcal{X}$ is denoted by $\mathrm{P}(\mathcal{X})$.

The relation between associated points and associated prime ideals of rings of sections is the best possible:

LEMMA 1.10: *Let $\mathcal{X}$ be an integral scheme and let $U \subseteq \mathcal{X}$ be an affine open subset. Then the equation $\mathrm{P}(\mathcal{X}) \cap U = \mathrm{P}(\mathcal{O}_{\mathcal{X}}(U))$ holds.*

Proof: For any domain A and any $p \in \mathrm{Spec}(A)$ one has $p \in \mathrm{P}(A) \Leftrightarrow pA_p \in \mathrm{P}(A_p)$ ([Mer], Prop. 5). ∎

As intended we can now identify a class of non-noetherian schemes for which $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ is a group:

THEOREM 1.11: *Let $\mathcal{X}$ be an integral scheme with coherent structure sheaf satisfying the condition:*

($\star$) *The local rings $\mathcal{O}_{\mathcal{X},x}$, $x \in \mathrm{P}(\mathcal{X})$, are valuation rings.*

Then:

1. *$\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ is a group.*
2. *For every affine open set $U \subseteq \mathcal{X}$ the ring $\mathcal{O}_{\mathcal{X}}(U)$ is a PvM-ring. In particular the scheme $\mathcal{X}$ is normal.*

Proof: One shows that any $\mathcal{J} \in \mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ satisfies $(\mathcal{O}_{\mathcal{X}} : \mathcal{J}) \in \mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ and the equation $[\mathcal{J}(\mathcal{O}_{\mathcal{X}} : \mathcal{J})]^{\wedge} = \mathcal{O}_{\mathcal{X}}$.

Coherence of $\mathcal{J}$ yields coherence of $(\mathcal{O}_{\mathcal{X}} : \mathcal{J})$. By Lemma 1.7 divisoriality of $(\mathcal{O}_{\mathcal{X}} : \mathcal{J})$ is equivalent to divisoriality of all stalks $(\mathcal{O}_{\mathcal{X}} : \mathcal{J})_x = (\mathcal{O}_{\mathcal{X},x} : \mathcal{J}_x)$, $x \in \mathcal{X}$. By assumption and Lemma 1.7 point 2 the stalks $\mathcal{J}_x$ are divisorial, hence $(\mathcal{O}_{\mathcal{X},x} : \mathcal{J}_x)$ is divisorial by a basic property of divisorial ideals in domains ([Gil], Th. 34.1 (3)).

Finitely generated fractionary ideals of a valuation domain are principal hence divisorial; using this fact for any $x \in \mathrm{P}(\mathcal{X})$ one has

$$([\mathcal{J}(\mathcal{O}_{\mathcal{X}} : \mathcal{J})]^{\wedge})_x = [\mathcal{J}_x(\mathcal{O}_{\mathcal{X},x} : \mathcal{J}_x)]^{\wedge} = \mathcal{J}_x(\mathcal{O}_{\mathcal{X},x} : \mathcal{J}_x) = \mathcal{O}_{\mathcal{X},x}.$$

Using Lemma 1.10 and (1.1 2.) this shows that for any affine open set $U \subseteq \mathcal{X}$ one has $[\mathcal{J}(\mathcal{O}_{\mathcal{X}} : \mathcal{J})]^{\wedge}(U) = [\mathcal{J}(U)(\mathcal{O}_{\mathcal{X}}(U) : \mathcal{J}(U))]^{\wedge} = \mathcal{O}_{\mathcal{X}}(U)$ and therefore $[\mathcal{J}(\mathcal{O}_{\mathcal{X}} : \mathcal{J})]^{\wedge} = \mathcal{O}_{\mathcal{X}}$.

The second statement follows from (1.3): By (A) at the beginning of this subsection the rings $\mathcal{O}_{\mathcal{X}}(U)$ are coherent. They are normal by condition ($\star$), Lemma 1.10 and (1.1 1). ∎

The condition $(\star)$ of Theorem 1.11 is the non-noetherian analogue of the condition of being *regular in codimension one* ([Ha], Ch. II - 6), which ensures a *good* divisor theory in the case of (separated) integral noetherian schemes. We could extend this analogy by using the valuation rings $\mathcal{O}_{\mathcal{X},x}$, $x \in \mathrm{P}(\mathcal{X})$, to obtain information about the structure of the group $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ as is done in the noetherian case. For the same reasons as in the affine case — redundance due to containment relations among these local rings — we will not follow that path. Instead we will use a set of valuation rings without this disadvantage.

Definition 1.12: $\mathrm{Val}(\mathcal{X}) := \{x \in \mathcal{X} \mid \mathcal{O}_{\mathcal{X},x}$ is a valuation ring$\}$, where $\mathcal{X}$ is an arbitrary scheme.

The set $\mathrm{Val}(\mathcal{X})$ is closed under generalization: For $x, y \in \mathcal{X}$ define $x \rightsquigarrow y$ as $y \in \overline{\{x\}}$ =Zariski closure of $\{x\}$. Localizations of valuation rings are valuation rings too; hence one finds that $y \in \mathrm{Val}(\mathcal{X})$ and $x \rightsquigarrow y$ implies $x \in \mathrm{Val}(\mathcal{X})$.

The spectrum of a valuation ring is totally ordered with respect to inclusion. As a consequence specialization gives $\mathrm{Val}(\mathcal{X})$ a treelike structure: For any $y \in \mathrm{Val}(\mathcal{X})$ the set $Y := \{x \in \mathrm{Val}(\mathcal{X}) \mid x \rightsquigarrow y\}$ has the property: For $x_1, x_2 \in Y$ either $x_1 \rightsquigarrow x_2$ or $x_2 \rightsquigarrow x_1$.

The valuation rings we are searching for to describe $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ are the maximal elements of $(\mathrm{Val}(\mathcal{X}), \rightsquigarrow)$ — provided they exist. Denote the set of maximal elements of $(\mathrm{Val}(\mathcal{X}), \rightsquigarrow)$ by $\mathrm{MaxVal}(\mathcal{X})$.

THEOREM 1.13: *Let $\mathcal{X}$ be a quasi-compact integral scheme with coherent structure sheaf, satisfying condition $(\star)$ of Theorem 1.11. Then every element $x \in \mathrm{Val}(\mathcal{X})$ specializes into a maximal element of* $(\mathrm{Val}(\mathcal{X}), \rightsquigarrow)$.

Proof: Quasi-compactness yields the existence of an affine open cover $\mathcal{X} = \bigcup_{i=1}^{r} U_i$. By Theorem 1.11 the rings $\mathcal{O}_{\mathcal{X}}(U_i)$, $i = 1, \ldots, r$, are PvM-rings.

Let $x \in \mathrm{Val}(\mathcal{X})$ be an arbitrary point, $x \in U_1$ say. Then by (1.5 3.) there exists $x_1 \in \mathrm{MaxVal}(U_1) \subseteq \mathrm{Val}(\mathcal{X})$ such that $x \rightsquigarrow x_1$. If $\overline{\{x_1\}} \cap \mathrm{Val}(\mathcal{X}) \neq \{x_1\}$ there exists $i \neq 1$ such that $\overline{\{x_1\}} \cap \mathrm{Val}(U_i) \neq \{x_1\}$. Again by (1.5 3.) one can therefore choose $x_2 \in \mathrm{MaxVal}(U_i)$ such that $x_1 \rightsquigarrow x_2$.

This procedure gives a specialization chain $x \rightsquigarrow x_1 \rightsquigarrow x_2 \rightsquigarrow \cdots \rightsquigarrow x_s$ with $s \leq r$ and $x_s \in \mathrm{MaxVal}(\mathcal{X})$. ∎

We have finally reached our aim of describing the structure of $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ using valuations of $F = K(\mathcal{X})$: For any $x \in \mathrm{MaxVal}(\mathcal{X})$ of the quasi-compact integral scheme $\mathcal{X}$ choose a valuation v of F such that $\mathcal{O}_v = \mathcal{O}_{\mathcal{X},x}$. Denote the family of valuations of F obtained in this way by $\mathcal{V}(\mathcal{X})$.

If the scheme $\mathcal{X}$ is separated, a valuation $v \in \mathcal{V}(\mathcal{X})$ uniquely determines a point $x \in \mathrm{MaxVal}(\mathcal{X})$ by the valuative criterion for separatedness ([EGA1], 1, 8.5.5).

Observe that for a fractional $\mathcal{O}_{\mathcal{X}}$-ideal of finite type every stalk $\mathcal{J}_x$, $x \in \mathrm{Val}(\mathcal{X})$, is a principal fractional ideal $a\mathcal{O}_{\mathcal{X},x}$. Given a valuation v of F with valuation ring $\mathcal{O}_{\mathcal{X},x}$ one can therefore define

$$v(\mathcal{J}) := v(a) = \min(v(b) |\, b \in \mathcal{J}_x).$$

THEOREM 1.14: *Let $\mathcal{X}$ be a separated, quasi-compact, integral scheme with coherent structure sheaf, satisfying condition $(\star)$ of Theorem 1.11. Then the assignment*

$$\mathrm{div}\colon \mathcal{D}_{\mathrm{coh}}(\mathcal{X}) \to \prod_{v \in \mathcal{V}(\mathcal{X})} vF, \mathcal{J} \mapsto (v(\mathcal{J}))_{v \in \mathcal{V}(\mathcal{X})}$$

is an injective group homomorphism.

Remark: div is in general not surjective; its image will be denoted by $\mathrm{Div}(\mathcal{X})$. According to our approach to Weil divisors $\mathrm{Div}(\mathcal{X})$ is the analogue to the group of Weil divisors on an integral, separated, noetherian, normal scheme.

Proof: The existence of the family $\mathcal{V}(\mathcal{X})$ follows from Theorem 1.13.

div is a homomorphism, since taking stalks in the points $x \in \mathrm{MaxVal}(\mathcal{X})$ turns the multiplication in $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ into (ordinary) multiplication of principal fractional ideals.

Injectivity: Let $\mathrm{div}(\mathcal{J}) = 0$ for some $\mathcal{J} \in \mathcal{D}_{\mathrm{coh}}(\mathcal{X})$. Since by condition $(\star)$ $\mathrm{P}(\mathcal{X}) \subseteq \mathrm{Val}(\mathcal{X})$ using Lemma 1.10 one obtains that for every open affine $U \subseteq \mathcal{X}$ the equations $\mathcal{J}_x = \mathcal{O}_{\mathcal{X},x}$, $x \in \mathrm{P}(\mathcal{O}_{\mathcal{X}}(U))$, hold. (1.1) yields $\mathcal{J}(U) = \mathcal{O}_{\mathcal{X}}(U)$, hence $\mathcal{J} = \mathcal{O}_{\mathcal{X}}$. ∎

The group $\prod_{v \in \mathcal{V}(\mathcal{X})} vF$ is partially ordered with the componentwise ordering. Furthermore, minima and maxima of finitely many elements exist in this group. Being a subgroup of $\prod_{v \in \mathcal{V}(\mathcal{X})} vF$ the group $\mathrm{Div}(\mathcal{X})$ is partially ordered too, and the relations

$$\mathrm{div}(\mathcal{I} \cap \mathcal{J}) = \max(\mathrm{div}(\mathcal{I}), \mathrm{div}(\mathcal{J})), \;\; \mathrm{div}((\mathcal{I} + \mathcal{J})^{\frown}) = \min(\mathrm{div}(\mathcal{I}), \mathrm{div}(\mathcal{J}))$$

show that $\mathrm{Div}(\mathcal{X})$ is closed under the formation of taking minima and maxima of finitely many elements. These relations are straightforward to prove and since we will not use them in this paper the proofs are omitted here.

2. Divisors on normal varieties over valuation domains

In this section we will apply the general results of the preceeding one to certain normal schemes $\mathcal{X}$ over an arbitrary valuation domain $\mathcal{O}_v$. These schemes are non-noetherian analogues of arithmetic varieties over a discrete valuation ring:

Definition 2.1: Let $\mathcal{O}_v$ be a valuation domain. A $\mathcal{O}_v$-scheme $\mathcal{X}$ is called a **$\mathcal{O}_v$-variety of relative dimension $n \in \mathbb{N}$** if it satisfies:

- $\mathcal{X}$ is integral, separated and of finite type over $\mathrm{Spec}(\mathcal{O}_v)$.
- The irreduzible components of all fibres $\mathcal{X} \times_{\mathcal{O}_v} \kappa(\mathfrak{p})$, $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_v)$, $\kappa(\mathfrak{p}) := \mathrm{Quot}(\mathcal{O}_v/\mathfrak{p})$, have dimension n.

A $\mathcal{O}_v$-variety $\mathcal{X}$ should be understood as a *totally ordered* family of noetherian equidimensional $\kappa(\mathfrak{p})$-schemes of finite type. The phrase *totally ordered* refers to the fact that the spectrum of a valuation ring is totally ordered with respect to inclusion. This fact is to a certain extent reflected by the specialization chains of points on $\mathcal{X}$.

Concerning the dimension of the fibres of a $\mathcal{O}_v$-variety, there is a nice result, due to M. Nagata, that simplifies our definition:

2.2 ([Nag], Lemma 2.1): *The dimensions of the irreducibel components of the fibres of an integral $\mathcal{O}_v$-scheme of finite type are all equal.*

As a consequence a $\mathcal{O}_v$-variety $\mathcal{X}$ has relative dimension n iff its generic fibre has dimension n. Nagata's result depends on the fact that an integral $\mathcal{O}_v$-algebra is flat. The latter holds due to the well-known equivalence of flatness and being torsionfree valid for modules over valuation domains. The flatness of an integral $\mathcal{O}_v$-algebra A has another consequence used at several points in this article: The Going-Down Theorem holds in the extension $A|\mathcal{O}_v$, which in particular implies that for every prime $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_v)$ every prime $q \in \mathrm{Spec}(A)$ minimal among the primes containing $\mathfrak{p}A$ lies over $\mathfrak{p}$.

In the sequel we will consider a fixed $\mathcal{O}_v$-variety $\mathcal{X}$ over the valuation domain $\mathcal{O}_v$ with maximal ideal $\mathcal{M}_v$, field of fractions K and corresponding valuation v of K.

Residue classes $a + \mathcal{M}_v$, $a \in \mathcal{O}_v$, will be denoted by av; consequently $Kv := \mathcal{O}_v/\mathcal{M}_v$.

To shorten notation the generic resp. closed fibre of $\mathcal{X}$ will be denoted by $X = \mathcal{X} \times_{\mathcal{O}_v} K$ resp. $Xv := \mathcal{X} \times_{\mathcal{O}_v} Kv$.

The investigations on the structure of $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ will involve the set $\mathcal{P}(X)$ of Weil prime divisors of the generic fibre X. If X is normal the set

$$X^{(1)} := \{x \in X|\ \dim(\mathcal{O}_{X,x}) = 1\}$$

precisely equals the set of generic points of prime divisors $P \in \mathcal{P}(X)$, and the local rings $\mathcal{O}_{X,x}$, $x \in X^{(1)}$, are discrete valuation rings. As usual the discrete normalized (i.e. with value group *equal* to $\mathbb{Z}$) valuation corresponding to $\mathcal{O}_{X,x}$ will then be denoted by v_P.

Observe also that if X is normal, one has the equality $X^{(1)} = \mathrm{P}(X)$, i.e. the associated points of X are precisely the generic points of Weil prime divisors.

The generic points $\mathrm{Gen}(\mathcal{X} \times_{\mathcal{O}_v} \kappa(\mathfrak{p})) := \{x \in \mathcal{X} \mid x \text{ generic point of } \mathcal{X} \times_{\mathcal{O}_v} \kappa(\mathfrak{p})\}$ of the other fibres will also play an important role: If $\mathcal{X}$ is normal the local rings $\mathcal{O}_{\mathcal{X},x}$, $x \in \mathrm{Gen}(\mathcal{X} \times_{\mathcal{O}_v} \kappa(\mathfrak{p}))$, are valuation rings too, as will be proved later in this section. The corresponding valuations of the function field $F = K(\mathcal{X})$ are of a special type called constant reductions:

Let $F|K$ be a finitely generated field extension of transcendence degree $r > 0$. A prolongation v of v to F is called a **constant reduction of $F|K$** if the residue field extension $F\mathrm{v}\,|Kv$ has transcendence degree r too.

A special class of constant reductions are the Gauß valuations: Consider the rational function field $F = K(x_1, \ldots, x_r)$ of transcendence degree r over K. The assignment

$$v_{\underline{x}}\Bigg(\sum_{(n_1,\ldots,n_r)\in\mathbb{N}^r} a_{(n_1,\ldots,n_r)} x_1^{n_1} \cdots x_r^{n_r}\Bigg) := \min\{v(a_{(n_1,\ldots,n_r)}) \mid (n_1, \ldots, n_r) \in \mathbb{N}^r\},$$

for polynomials $\sum_{(n_1,\ldots,n_r)} a_{(n_1,\ldots,n_r)} x_1^{n_1} \cdots x_r^{n_r} \in K[x_1, \ldots, x_r]$ extends to a valuation of F prolonging v such that the elements $x_1 v_{\underline{x}}, \ldots, x_r v_{\underline{x}}$ are algebraically independent over Kv and hence $F v_{\underline{x}} = Kv(x_1 v_{\underline{x}}, \ldots, x_r v_{\underline{x}})$ is the rational function field of transcendence degree r over Kv; $v_{\underline{x}}$ is called the **Gauß prolongation of v with respect to $\underline{x} := (x_1, \ldots, x_r)$.**

It is well-known that any constant reduction v of a finitely generated field extension $F|K$ prolonging a given valuation v on K is a prolongation of a Gauß valuation $v_{\underline{x}}$ for a suitable transcendence base $\underline{x} = (x_1, \ldots, x_r)$ of $F|K$.

For every $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_v)$ the valuation corresponding to the valuation ring $(\mathcal{O}_v)_\mathfrak{p}$ is denoted by $v_\mathfrak{p}$. The fibre product $\mathcal{X}_\mathfrak{p} := \mathcal{X} \times_{\mathcal{O}_v} (\mathcal{O}_v)_\mathfrak{p} = \mathcal{X} \times_{\mathcal{O}_v} \mathcal{O}_{v_\mathfrak{p}}$ is a $\mathcal{O}_{v_\mathfrak{p}}$-variety. For any $x \in \mathcal{X} \times_{\mathcal{O}_v} \kappa(\mathfrak{p})$ one has $\mathcal{O}_{\mathcal{X},x} = \mathcal{O}_{\mathcal{X}_\mathfrak{p},x}$.

2.1 THE STRUCTURE OF THE GROUP OF (GENERALIZED) WEIL DIVISORS. The first step in studying $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ of a normal $\mathcal{O}_v$-variety $\mathcal{X}$ is to verify that it satisfies the condition $(\star)$ introduced in Theorem 1.11. We therefore have to deal with coherence of the structure sheaf of such a scheme.

The finiteness properties of algebras over a valuation ring $\mathcal{O}_v$ have been investigated by several authors. We collect the relevant results for the present article in the following statement:

2.3 ([Sab], Prop. 3, [Nag], Th. 3): *Let $\mathcal{O}_v$ be a valuation ring.*

1. *The polynomial ring $\mathcal{O}_v[X_1,\ldots,X_n]$ in n variables is coherent.*
2. *Every flat finitely generated $\mathcal{O}_v$-algebra A is of finite presentation.*

Consequently we get:

THEOREM 2.4: *The structure sheaf of an integral $\mathcal{O}_v$-scheme of finite type is coherent.*

Proof: Since coherence is a local property it suffices to treat the affine case $\mathcal{X} = \operatorname{Spec}(A)$, A a finitely generated integral $\mathcal{O}_v$-algebra. By (A) at the beginning of the preceding section coherence of $\mathcal{O}_\mathcal{X}$ is then equivalent to the coherence of the ring A. Now by (2.3 2.) an integral finitely generated $\mathcal{O}_v$-algebra is of finite presentation. Therefore $A = \mathcal{O}_v[X_1,\ldots,X_n]/I$, where I is a finitely generated prime ideal of the polynomial ring $\mathcal{O}_v[X_1,\ldots,X_n]$.

By (2.3 1.) the polynomial ring $\mathcal{O}_v[X_1,\ldots,X_n]$ is coherent and by a basic property of coherent rings ([Gla], Th. 2.4.1) A is therefore coherent too. ∎

COROLLARY 2.5: *Any* normal *$\mathcal{O}_v$-variety $\mathcal{X}$ satisfies condition ($\star$); in particular, $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ of such a scheme is a group.*

Proof: Theorem 2.4 implies that a normal $\mathcal{O}_v$-variety can be covered by spectra of normal coherent domains, which by (1.3) of subsection 1.1 are PvM-rings. The assertion now follows from (1.2) and Lemma 1.10. ∎

By definition a $\mathcal{O}_v$-variety is quasi-compact, hence Theorem 1.14 shows that $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ of a normal $\mathcal{O}_v$-variety possesses a description in terms of the family $\mathcal{V}(\mathcal{X})$ introduced in subsection 1.2. The next step is therefore to determine these valuations resp. investigate the set $\operatorname{Val}(\mathcal{X})$.

THEOREM 2.6: *Let $\mathcal{X}$ be a normal $\mathcal{O}_v$-variety with field of rational functions F; then the following results hold:*

1. $\operatorname{Val}(\mathcal{X}) = X^{(1)} \cup \bigcup_{\mathcal{P}\in\operatorname{Spec}(\mathcal{O}_v)\smallsetminus 0} \operatorname{Gen}(\mathcal{X}\times_{\mathcal{O}_v}\kappa(\mathcal{P}))$.
2. $\{\mathcal{O}_{\mathcal{X},x} \mid x\in X^{(1)}\} = \{\mathcal{O}_{v_P} \mid P\in\mathcal{P}(X)\}$; *in particular these local rings are discrete valuation rings.*
3. *For every $\mathcal{P}\in\operatorname{Spec}(\mathcal{O}_v)$ the local rings $\mathcal{O}_{\mathcal{X},x}$, $x\in\operatorname{Gen}(\mathcal{X}\times_{\mathcal{O}_v}\kappa(\mathcal{P}))$, are valuation rings of constant reductions of $F|K$ prolonging $v_\mathcal{P}$.*

The proof of the main point 1 of this theorem is based on an application of the following version of Zariski's Main Theorem:

2.7 ([Pes]): *Let A be a finitely generated R-algebra. Let $q \in \operatorname{Spec}(A)$ be a prime ideal satisfying:*

- *q is minimal and maximal among the primes of A lying over $q \cap R$.*

Then there exists an R-algebra $A' \subseteq A$ and an element $t \in A' \setminus q \cap A'$ such that $A'|R$ is finite and $A'_t = A_t$ holds.

We will apply (2.7) to prove:

LEMMA 2.8: *Let A be a finitely generated $\mathcal{O}_v$-domain. Let $q \in \operatorname{Spec}(A)$ be such that $q \cap \mathcal{O}_v = \mathcal{P} \neq 0$ and q is* not *minimal among the primes containing $\mathcal{P}A$. Then* $\operatorname{Spec}(A_q)$ *is* not *totally ordered by inclusion; in particular, the local ring A_q is* no *valuation ring.*

Proof: It suffices to prove the lemma for the case $\mathcal{P} = \mathcal{M}_v$: Otherwise replace A by $A \otimes_{\mathcal{O}_v} (\mathcal{O}_v)_{\mathcal{P}}$.

One first treats the case of a polynomial ring $A = \mathcal{O}_v[x_1, \ldots, x_n]$: $\mathcal{M}_v A = \mathcal{M}_v[x_1, \ldots, x_n]$ is then a prime ideal and $A/\mathcal{M}_v A = Kv[\overline{x_1}, \ldots, \overline{x_n}]$ is the polynomial ring in n variables over Kv.

Let q be a prime ideal properly containing $\mathcal{M}_v[x_1, \ldots, x_n]$. The prime ideal $q/\mathcal{M}_v[x_1, \ldots, x_n]$ of the polynomial ring $Kv[\overline{x_1}, \ldots, \overline{x_n}]$ contains a monic prime polynomial $\overline{f}$. By Gauß' lemma any $f \in q$ such that $f + \mathcal{M}_v[x_1, \ldots, x_n] = \overline{f}$ is a prime polynomial; the prime ideal $f\mathcal{O}_v[x_1, \ldots, x_n]$ is of height 1. Hence q contains infinitely many prime ideals of height 1. Since these prime ideals are incomparable with respect to inclusion the spectrum of A_q is not totally ordered.

Now one can prove the lemma in the general case: Let A be a finitely generated $\mathcal{O}_v$-algebra and assume A_q to be a valuation ring, where q is a prime ideal that is *not* minimal among the primes containing $\mathcal{M}_v A$. One shows that these assumptions lead to a contradiction.

By assumption about q and since $\operatorname{Spec}(A_q)$ is totally ordered, there exists a unique prime $p \subset q$ such that p is minimal among the primes containing $\mathcal{M}_v A$. The ring A_q/pA_q is a valuation ring and a localization of the finitely generated Kv-algebra A/p. It follows that A_q/pA_q is a discrete valuation ring and thus that there exists no prime between p and q.

Let $\overline{x_1}, \ldots, \overline{x_n} \in A/p$ be Kv-algebraically independent elements such that the extension $A/p|Kv[\overline{x_1}, \ldots, \overline{x_n}]$ is finite. Choose foreimages $x_1, \ldots, x_n \in A$ of these elements; then $\mathcal{O}_v[x_1, \ldots, x_n]$ is a polynomial ring and by construction the equation $p \cap \mathcal{O}_v[x_1, \ldots, x_n] = \mathcal{M}_v[x_1, \ldots, x_n]$ holds.

q is minimal and maximal among the primes of A lying over $q_0 := q \cap \mathcal{O}_v[x_1, \ldots, x_n]$: If q were not maximal among those primes, q/p would not be maximal among the primes of A/p lying over $\overline{q_0} := q_0/\mathcal{M}_v[x_1,\ldots,x_n]$, which contradicts the finiteness of $A/p|Kv[\overline{x_1},\ldots,\overline{x_n}]$.

Let $q_1 \subset q$ be a prime lying over q_0. Since the primes contained in q are totally ordered by inclusion one has $q_1 \subset p$ or $p \subset q_1$. Since there are no primes properly lying between p and q the second possibility cannot occur. On the other hand, q_0 contains $\mathcal{M}_v[x_1,\ldots,x_n]$, hence q_1 contains $\mathcal{M}_vA$. Since p has been chosen to be minimal among the primes containing $\mathcal{M}_vA$, the inclusion $q_1 \subset p$ is impossible too.

One can now apply (2.7) to the ring extension $A|\mathcal{O}_v[x_1,\ldots,x_n]$: There exists a $\mathcal{O}_v[x_1,\ldots,x_n]$-algebra A' finite over $\mathcal{O}_v[x_1,\ldots,x_n]$ and an element $t \in A' \setminus q'$, $q' := q \cap A'$, such that $A'_t = A_t$. In particular, one has $A_q = A'_{q'}$ and therefore that $\mathrm{Spec}(A'_{q'})$ is totally ordered. The Going Down Theorem in the finite extension $A'|\mathcal{O}_v[x_1,\ldots,x_n]$ now yields that $\mathrm{Spec}(\mathcal{O}_v[x_1,\ldots,x_n]_{q_0})$ is totally ordered.

On the other hand, q_0 is properly containing $\mathcal{M}_v[x_1,\ldots,x_n]$: It has already been shown above that $\mathrm{height}(q/p) = 1$ in A/p; again using finiteness $\mathrm{height}(\overline{q_0}) = 1$ too, hence q_0 is properly containing $\mathcal{M}_v[x_1,\ldots,x_n]$. But then it follows from the first part of this proof that $\mathrm{Spec}(\mathcal{O}_v[x_1,\ldots,x_n]_{q_0})$ is not totally ordered — the desired contradiction. ∎

Proof of Theorem 2.6: Since X is dense in $\mathcal{X}$ one has $\mathcal{O}_{\mathcal{X},x} = \mathcal{O}_{X,x}$ for any $x \in X^{(1)}$. The latter local ring is a discrete valuation ring since X is a normal K-variety.

Let $x \in \mathrm{Gen}(\mathcal{X} \times_{\mathcal{O}_v} \kappa(\mathcal{P}))$; without loss of generality one can assume that $\mathcal{P} = \mathcal{M}_v$ holds: Otherwise take the localization $\mathcal{X} \times_{\mathcal{O}_v} \mathcal{O}_{v_\mathcal{P}}$, which is a normal $\mathcal{O}_{v_\mathcal{P}}$-variety such that x lies on its closed fibre.

Choose an affine open neighborhood $U \subseteq \mathcal{X}$ of x; the $\mathcal{O}_v$-algebra $A := \mathcal{O}_\mathcal{X}(U)$ is then finitely generated. Let $q \lhd A$ be the prime ideal corresponding to x.

By definition of an $\mathcal{O}_v$-variety and the choice of x one has

$$\mathrm{trdeg}(A/q|Kv) = \dim(A/q) = \dim(A \otimes_{\mathcal{O}_v} K) = \mathrm{trdeg}(F|K).$$

Therefore there exists a transcendence basis $x_1,\ldots,x_r \in A$ of $F|K$ such that $x_1+q,\ldots,x_r+q$ is a transcendence basis of $\mathrm{Quot}(A/q)|Kv$.

Working in the ring extension $A|\mathcal{O}_v[x_1,\ldots,x_r]$ one obtains

$$q \cap \mathcal{O}_v[x_1,\ldots,x_r] = \mathcal{M}_v[x_1,\ldots,x_r] \tag{2}$$

by the choice of the elements x_i, $i=1,\ldots,r$. By definition of the Gauß valuation the rings $\mathcal{O}_{v_{\underline{x}}}$ and $\mathcal{O}_v[x_1,\ldots,x_r]_{\mathcal{M}_v[x_1,\ldots,x_r]}$ are equal, hence the normal ring A_q

contains the integral closure $\mathcal{O}$ of $\mathcal{O}_{v_{\underline{x}}}$ in F. The ring $\mathcal{O}$ is a Prüfer ring, therefore A_q is a valuation ring $\mathcal{O}_{\mathrm{v}}$, which by (2) prolongs the Gauß valuation $v_{\underline{x}}$ and therefore also v. Moreover, the residue field extension $F\mathrm{v}\,|Kv$ has transcendence degree r, thus v is a constant reduction of $F|K$ as asserted.

The points 2 and 3 are now completely proved and to verify point 1 it remains to show that for any $x \in \mathcal{X}$ such that $x \notin X^{(1)} \cup \bigcup_{\mathcal{P}\in\mathrm{Spec}(\mathcal{O}_v)\setminus 0} \mathrm{Gen}(\mathcal{X}\times_{\mathcal{O}_v}\kappa(\mathcal{P}))$ the local ring $\mathcal{O}_{\mathcal{X},x}$ is *no* valuation ring. For the points $x\in\mathcal{X}$ lying on the generic fibre this is obvious, since the ring $\mathcal{O}_{\mathcal{X},x}$ is then noetherian.

For a point $x \in \mathcal{X}\times_{\mathcal{O}_v}\kappa(\mathcal{P})\setminus \mathrm{Gen}(\mathcal{X}\times_{\mathcal{O}_v}\kappa(\mathcal{P}))$ choosing an affine open neighborhood U of x gives a prime ideal $q \in \mathrm{Spec}(\mathcal{O}_{\mathcal{X}}(U))$, which is not minimal among the primes containing $\mathcal{P}\mathcal{O}_{\mathcal{X}}(U)$. Applying Lemma 2.8 now shows the assertion. ∎

Let $\mathrm{V}_{\mathcal{P}}$ denote the finite set of constant reductions of the function field F of $\mathcal{X}$ defined by the fibre $\mathcal{X}\times_{\mathcal{O}_v}\kappa(\mathcal{P})$, $\mathcal{P}\in\mathrm{Spec}(\mathcal{O}_v)\setminus 0$, according to point 3 of the preceding theorem. These sets occur in the valuation theoretic description of $\mathrm{Div}(\mathcal{X})$; as we will see the set $\mathrm{V} := \mathrm{V}_{\mathcal{M}_v}$ is of particular interest.

Proposition 2.9: *A normal $\mathcal{O}_v$-variety $\mathcal{X}$ of relative dimension n has the properties:*

1. *There exist subsets $T_{\mathcal{P}} \subseteq \mathrm{Gen}(\mathcal{X}\times_{\mathcal{O}_v}\kappa(\mathcal{P}))$, $\mathcal{P}\in\mathrm{Spec}(\mathcal{O}_v)\setminus\{0,\mathcal{M}_v\}$, such that*

$$\mathrm{MaxVal}(\mathcal{X}) = X^{(1)} \cup \left(\bigcup_{\mathcal{P}\in\mathrm{Spec}(\mathcal{O}_v)\setminus\{0,\mathcal{M}_v\}} T_{\mathcal{P}}\right) \cup \mathrm{Gen}(Xv).$$

Let $\mathrm{V}'_{\mathcal{P}} \subseteq \mathrm{V}_{\mathcal{P}}$, $\mathcal{P}\in\mathrm{Spec}(\mathcal{O}_v)\setminus\{0,\mathcal{M}_v\}$, be the subsets of $\mathrm{V}_{\mathcal{P}}$ defined through $\{\mathcal{O}_{\mathrm{v}_{\mathcal{P}}} |\ \mathrm{v}_{\mathcal{P}}\in\mathrm{V}'_{\mathcal{P}}\} = \{\mathcal{O}_{\mathcal{X},x} |\ x\in T_{\mathcal{P}}\}$; then

$$\mathcal{V}(\mathcal{X}) = \{v_P |\ P\in\mathcal{P}(X)\} \cup \left(\bigcup_{\mathcal{P}\in\mathrm{Spec}(\mathcal{O}_v)\setminus\{0,\mathcal{M}_v\}} \mathrm{V}'_{\mathcal{P}}\right) \cup \mathrm{V},$$

and therefore the div*-homomorphism introduced in Theorem 1.14 is an embedding*

$$\mathrm{div}: \mathcal{D}_{\mathrm{coh}}(\mathcal{X}) \to \mathrm{Div}(X) \oplus \prod_{\mathcal{P}\in\mathrm{Spec}(\mathcal{O}_v)\setminus\{0,\mathcal{M}_v\}} \left(\bigoplus_{\mathrm{v}_{\mathcal{P}}\in\mathrm{V}'_{\mathcal{P}}} \mathrm{v}_{\mathcal{P}}\,F\right) \oplus \bigoplus_{\mathrm{v}\in\mathrm{V}} \mathrm{v}\,F.$$

2. *If the structure morphism of $\mathcal{X}|\mathcal{O}_v$ is closed the sets $T_{\mathcal{P}}$ are empty, hence the* div*-homomorphism becomes*

$$\mathrm{div}: \mathcal{D}_{\mathrm{coh}}(\mathcal{X}) \to \mathrm{Div}(X) \oplus \bigoplus_{\mathrm{v}\in\mathrm{V}} \mathrm{v}\,F.$$

Proof: A $\mathcal{O}_v$-variety $\mathcal{X}$ is by definition quasi-compact, which by Theorem 1.13 implies the existence of $\mathrm{MaxVal}(\mathcal{X})$. The existence of the sets $T_{\mathcal{P}}$ now follows from Theorem 2.6 point 1. The same theorem shows the inclusion $\mathrm{Gen}(Xv) \subseteq \mathrm{MaxVal}(\mathcal{X})$.

Furthermore, one always has the inclusion $X^{(1)} \subseteq \mathrm{MaxVal}(\mathcal{X})$: Assume $x \rightsquigarrow y$ for $x \in X^{(1)}$, $y \in \mathrm{Val}(\mathcal{X})$, $y \neq x$. Since x is the generic point of a prime divisor on the generic fibre X of $\mathcal{X}$ one has $\mathrm{trdeg}(\kappa(x)|K) = \dim(X) - 1 = n - 1$. On the other hand, Theorem 2.6 point 1 shows that $y \in \mathrm{Gen}(\mathcal{X} \times_{\mathcal{O}_v} \kappa(\mathcal{P}))$ for $\mathcal{P} \neq 0$ holds; thus $\mathrm{trdeg}(\kappa(y)|\kappa(\mathcal{P})) = n$. The assumption $x \rightsquigarrow y$ consequently leads to the contradiction $n - 1 \geq n$.

The assertion about $\mathcal{V}(\mathcal{X})$ and the div-homomorphism in the general case are now clear.

Assume next that $\mathcal{X}$ has a closed structure morphism ϕ: $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_v)$. For any $x \in \mathrm{Gen}(\mathcal{X} \times_{\mathcal{O}_v} \kappa(\mathcal{P}))$ the set $\phi(\overline{\{x\}})$ is by assumption closed, hence contains $\mathcal{M}_v$. Therefore there exists a point $y \in Xv \cap \overline{\{x\}}$; let U be an affine open neighborhood of y and $A := \mathcal{O}_{\mathcal{X}}(U)$. Let q_x resp. q_y be the primes of A corresponding to x resp. y, so that $q_x \subseteq q_y$ holds. $\overline{A} := A/q_x$ is a finitely generated algebra over the valuation ring $\overline{\mathcal{O}_v} := \mathcal{O}_v/\mathcal{P}$ with maximal ideal $\overline{\mathcal{M}_v}$.

Let $\overline{q} \lhd \overline{A}$ be a prime ideal minimal over $\overline{\mathcal{M}_v}\overline{A}$. Then by [Nag], Lemma 2.1 and the choice of x one has

$$\mathrm{trdeg}(\overline{A}/\overline{q}|\overline{\mathcal{O}_v}/\overline{\mathcal{M}_v}) = \mathrm{trdeg}(\overline{A}|\overline{\mathcal{O}_v}) = n.$$

Hence there exists a prime $q \lhd A$ with the properties

$$q_x \subseteq q \subseteq q_y, \mathrm{trdeg}(A/q|Kv) = n.$$

The latter implies $x' \in \mathrm{Gen}(Xv)$, denoting by $x' \in \mathcal{X}$ the point corresponding to q, while the first property shows $x \rightsquigarrow x'$. This shows that x is not maximal with respect to specialization among the points in $\bigcup_{\mathcal{P} \in \mathrm{Spec}(\mathcal{O}_v) \setminus 0} \mathrm{Gen}(\mathcal{X} \times_{\mathcal{O}_v} \kappa(\mathcal{P}))$ unless $x \in Xv$, which proves the assertion. ∎

Next, we have to investigate the image $\mathrm{Div}(\mathcal{X})$ of the homomorphism div. This is done by taking a closer look at those subschemes of $\mathcal{X}$ obtained by taking the Zariski closure of prime divisors of the generic fibre of $\mathcal{X}$: Let $P \subset X$ be a Weil prime divisor of X and denote by $\mathbf{P} \subset \mathcal{X}$ the Zariski closure of P on $\mathcal{X}$. $\mathbf{P}$ is then a closed irreducible subscheme of $\mathcal{X}$ carrying the induced reduced subscheme structure. The $\mathcal{O}_{\mathcal{X}}$-ideal defining the reduced subscheme structure on $\mathbf{P}$ will in the sequel be denoted by $\mathcal{J}_P \subset \mathcal{O}_{\mathcal{X}}$.

THEOREM 2.10: *Let $\mathcal{X}$ be a normal $\mathcal{O}_v$-variety of relative dimension n; then for any prime divisor $P \in \mathcal{P}(X)$:*

1. *$\mathcal{J}_P$ is a coherent divisorial $\mathcal{O}_{\mathcal{X}}$-ideal: $\mathcal{J}_P \in \mathcal{D}_{\mathrm{coh}}(\mathcal{X})$.*
2. *$\mathbf{P}$ is a $\mathcal{O}_v$-variety of relative dimension $n-1$.*
3. *The map $\mathcal{P}(X) \to \mathcal{D}_{\mathrm{coh}}(\mathcal{X}), P \mapsto \mathcal{J}_P$ extends to an injective group homomorphism t: $\mathrm{Div}(X) \to \mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ with the property*

$$\forall D \in \mathrm{Div}(X)\colon \mathrm{div}(t(D)) = D \oplus 0 \quad \textit{(see Proposition 2.9).}$$

Proof: 1. Let $U \subseteq \mathcal{X}$ be an affine open set such that $\mathbf{P} \cap U \neq \emptyset$; then $p := \mathcal{J}_P(U)$ is a prime ideal with $p \cap \mathcal{O}_v = 0$. Since $\mathcal{O}_{\mathcal{X}}(U)$ is finitely generated over $\mathcal{O}_v$ one has $\mathcal{O}_{\mathcal{X}}(U) = \mathcal{O}_v[x_1, \ldots, x_n]/I$ and $p = q/I$ with a prime q of the polynomial ring $\mathcal{O}_v[x_1, \ldots, x_n]$ lying over 0. By [Gla], Theorem 7.4.3 the prime q and hence p is finitely generated. Since $\mathcal{O}_{\mathcal{X}}(U)$ is a coherent ring p is a coherent $\mathcal{O}_{\mathcal{X}}(U)$-ideal, and consequently $\mathcal{J}_P$ is a coherent sheaf.

By Theorem 2.6 point 2, $\mathcal{O}_{\mathcal{X}}(U)_p$ is a (discrete) valuation ring, hence $p \in \mathrm{Val}(\mathcal{O}_{\mathcal{X}}(U))$. Applying (1.3) and (1.5 3.) yields $p \in \mathcal{D}^t(\mathcal{O}_{\mathcal{X}}(U))$, which shows divisoriality of p because finitely generated t-ideals are divisorial.

2. It only remains to check the dimensions of the fibres of $\mathbf{P}|\mathcal{O}_v$. Since the generic fibre of $\mathbf{P}$ is the prime divisor $P \subset X$, which has dimension $n-1$, the assertion follows from (2.2).

3. It suffices to prove $\mathrm{div}(\mathcal{J}_P) = P \oplus 0$. Let $y \in \mathcal{X}$ be the generic point of $\mathbf{P}$; one has to check the relations $(\mathcal{J}_P)_x = \mathcal{O}_{\mathcal{X},x}$ for all $x \in \mathrm{MaxVal}(\mathcal{X}) \setminus \{y\}$ and $(\mathcal{J}_P)_y = \mathcal{M}_{\mathcal{X},y}$. The latter holds by definition of $\mathbf{P}$; furthermore, it is clear that $(\mathcal{J}_P)_x = \mathcal{O}_{\mathcal{X},x}$ for all $x \in X^{(1)}$. Finally, the relation $\mathbf{P} \cap \mathrm{Gen}(\mathcal{X} \times_{\mathcal{O}_v} \kappa(\mathfrak{p})) \neq \emptyset$ would contradict point 1 of Proposition 2.9, which shows $(\mathcal{J}_P)_x = \mathcal{O}_{\mathcal{X},x}$ for any $x \in \mathrm{Gen}(\mathcal{X} \times_{\mathcal{O}_v} \kappa(\mathfrak{p}))$. ∎

The next result is for aesthetic reasons only formulated for the case of a normal $\mathcal{O}_v$-variety with *closed* structure morphism, but it holds for general normal $\mathcal{O}_v$-varieties with obvious modifications:

COROLLARY 2.11: *Let $\mathcal{X}$ be a normal $\mathcal{O}_v$-variety with closed structure morphism. Then there exists a subgroup $\mathrm{Ver}(\mathcal{X}) \leq \bigoplus_{\mathrm{v} \in \mathrm{V}} \mathrm{v}\, F$ such that*

$$\mathrm{Div}(\mathcal{X}) = \mathrm{Div}(X) \oplus \mathrm{Ver}(\mathcal{X}).$$

Proof: Point 3 of Theorem 2.10 shows that $\mathrm{Div}(X) \oplus 0 \subseteq \mathrm{Div}(X) \oplus \bigoplus_{\mathrm{v} \in \mathrm{V}} \mathrm{v}\, F$ is actually a subgroup of $\mathrm{Div}(\mathcal{X})$, which proves the assertion. ∎

The elements of $\mathrm{Ver}(\mathcal{X})$ are for obvious reasons called **vertical divisors**. Observe that in contrast to the noetherian case neither the closed fibre of $\mathcal{X}$ nor its irreducible components need to be (vertical) divisors, since the corresponding ideal sheaves are in general not coherent.

The elements of the subgroup $\mathrm{Div}(X) \oplus 0$ are called **horizontal divisors.**

As already remarked in the affine case, the group $\mathrm{Ver}(\mathcal{X})$ will in general not be equal to $\bigoplus_{\mathrm{v}\in\mathrm{V}} \mathrm{v}\, F$. We will close this section with a result that sheds some light on the structure of $\mathrm{Ver}(\mathcal{X})$: Define

$$\mathcal{O}_{\mathrm{V}} := \bigcap_{\mathrm{v}\in\mathrm{V}} \mathcal{O}_{\mathrm{v}};$$

this ring is a semilocal Prüfer domain ([Kap], Th. 107). We will make use of the following well-known properties of Prüfer domains:

- A Prüfer domain is a coherent PvM-ring, since its finitely generated fractionary ideals are invertible.
- In a Prüfer domain A the relation $\mathrm{MaxVal}(A) = \mathrm{MaxSpec}(A)$ holds.

Recall furthermore that invertible ideals of a semilocal ring are principal.

Combining these facts one concludes $\mathcal{V}(\mathcal{O}_{\mathrm{V}}) = \mathrm{V}$ and from (1.4) in subsection 1.1 that $\mathrm{Div}(\mathcal{O}_{\mathrm{V}})$ is a group, which is explicitly given by

$$\mathrm{Div}(\mathcal{O}_{\mathrm{V}}) = \{(\mathrm{v}(a))_{\mathrm{v}\in\mathrm{V}} \mid a \in F^*\}.$$

Again the following result can easily be modified in such a way that one can drop the closedness assumption on the structure morphism.

PROPOSITION 2.12: *Let $\mathcal{X}$ be a normal $\mathcal{O}_v$-variety with closed structure morphism. Then:*

1. $\mathrm{Div}(\mathcal{O}_{\mathrm{V}}) \leq \mathrm{Ver}(\mathcal{X})$.
2. *If* $\mathrm{Gen}(Xv)$ *posesses an affine open neighborhood then* $\mathrm{Ver}(\mathcal{X}) = \mathrm{Div}(\mathcal{O}_{\mathrm{V}})$.
3. *If* $\dim(\mathcal{O}_{\mathrm{v}}) = 1$ *for all* $\mathrm{v} \in \mathrm{V}$ *then* $\mathrm{Div}(\mathcal{O}_{\mathrm{V}}) = \mathrm{Ver}(\mathcal{X}) = \bigoplus_{\mathrm{v}\in\mathrm{V}} \mathrm{v}\, F$.

Proof: 1. Any divisor $D \in \mathrm{Div}(\mathcal{O}_{\mathrm{V}})$ of the PvM-ring $\mathcal{O}_{\mathrm{V}}$ is defined by a principal fractionary ideal $a\mathcal{O}_{\mathrm{V}}$, $a \in F$. The principal ideal sheaf $a\mathcal{O}_{\mathcal{X}}$ then defines the divisor $\mathrm{div}(a\mathcal{O}_{\mathcal{X}}) = D' + D \in \mathrm{Div}(\mathcal{X})$ with $D' \in \mathrm{Div}(X)$. According to Theorem 2.10 there exists $\mathcal{J} \in \mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ such that $\mathrm{div}(\mathcal{J}) = D'$. Hence $\mathrm{div}((a\mathcal{O}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}} : \mathcal{J}))^\frown) = D$, which proves the assertion.

2. Let $\mathcal{J} \in \mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ be such that $\mathrm{div}(\mathcal{J}) \in \mathrm{Ver}(\mathcal{X})$. Let U be an affine open neigborhood of $\mathrm{Gen}(Xv)$. Then $\mathcal{J}(U)$ is a finitely generated fractionary $\mathcal{O}_{\mathcal{X}}(U)$-ideal and $\mathcal{O}_{\mathcal{X}}(U) \subset \mathcal{O}_{\mathrm{V}}$. Hence $\mathcal{J}(U)\mathcal{O}_{\mathrm{V}}$ is a finitely generated thus invertible $\mathcal{O}_{\mathrm{V}}$-ideal, which by definition satisfies $\mathrm{v}(\mathcal{J}(U)\mathcal{O}_{\mathrm{V}}) = \mathrm{v}(\mathcal{J})$ for all $\mathrm{v} \in \mathrm{V}$.

3. By the approximation theorem [End], (11.17) the equality $\mathrm{Div}(\mathcal{O}_{\mathrm{V}}) = \bigoplus_{\mathrm{v}\in\mathrm{V}} \mathrm{v}\, F$ holds, which proves the assertion. ∎

2.2 THE LOCAL WEIL DIVISOR GROUPS AND CARTIER DIVISORS. Let $\mathcal{X}$ be an integral scheme with coherent structure sheaf and function field F. The group $\mathrm{CaDiv}(\mathcal{X})$ of Cartier divisors on $\mathcal{X}$ can be identified with the group of invertible $\mathcal{O}_{\mathcal{X}}$-submodules of the constant sheaf $\underline{F}$ ([Ha], Ch. II, Prop. 6.13). This identification fits the purposes of the present subsection, for which reason we always view Cartier divisors as invertible $\mathcal{O}_{\mathcal{X}}$-ideals. Since invertible $\mathcal{O}_{\mathcal{X}}$-ideals are divisorial and of finite type, $\mathrm{CaDiv}(\mathcal{X})$ is a subgroup of the semigroup $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$.

As in the noetherian case those integral schemes having the property that $\mathrm{CaDiv}(\mathcal{X}) = \mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ are of particular interest. The extent to which $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ differs from $\mathrm{CaDiv}(\mathcal{X})$ is measured by the class semigroups of the local rings of $\mathcal{X}$. In this subsection we will therefore study the structure of the local Weil divisor groups $\mathrm{Div}(\mathcal{O}_{\mathcal{X},x})$ in the points of a normal $\mathcal{O}_v$-variety $\mathcal{X}$ and give a criterion for $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ to be equal to $\mathrm{CaDiv}(\mathcal{X})$.

Let A be an integral domain; denote by $\mathcal{H}(A)$ the multiplicative group of non-zero principal fractionary A-ideals. $\mathcal{H}(A)$ is a subgroup of the semigroup $\mathcal{D}^v_{\mathrm{fin}}(A)$ and one defines the **class semigroup of** A as $\mathrm{Cl}(A) := \mathcal{D}^v_{\mathrm{fin}}(A)/\mathcal{H}(A)$.

We are interested in the case $\mathrm{Cl}(A) = 0$; in that case $\mathcal{D}^v_{\mathrm{fin}}(A) = \mathcal{H}(A)$ is a group, that is, A is a PvM-ring. If, in addition, A is noetherian or, more general, a Krull ring then it is factorial.

We will furthermore need the following simple result on the behavior of $\mathcal{D}^v_{\mathrm{fin}}(A)$ in flat extensions of A:

LEMMA 2.13: *Let $B|A$ be a flat extension of coherent domains. Then:*

1. *The map ℓ: $\mathcal{D}^v_{\mathrm{fin}}(A) \to \mathcal{D}^v_{\mathrm{fin}}(B), I \mapsto IB$ is welldefined and a group homomorphism.*
2. *If $\mathrm{Quot}(A) = \mathrm{Quot}(B)$ the homomorphism ℓ is surjective.*

Proof: The coherence of A implies that any $I \in \mathcal{D}^v_{\mathrm{fin}}(A)$ is finitely generated. This fact and the flatness of $B|A$ yields $(A : I)B = (B : IB)$ ([Mat], (3.H.)). Since $(A : I)$ is finitely generated too, one gets $IB = \widehat{I}B = \widehat{IB}$, proving point 1.

Any $J \in \mathcal{D}^v_{\mathrm{fin}}(B)$ is finitely generated: $J = \sum_{i=1}^r Bb_i$. The fractionary ideal $I := (\sum_{i=1}^r Ab_i)^\frown \in \mathcal{D}^v_{\mathrm{fin}}(A)$ satisfies $IB = \widehat{J} = J$. ∎

We start with a general result ensuring equality of Weil and Cartier divisors:

PROPOSITION 2.14:

1. *Let $\mathcal{X}$ be an integral scheme with coherent structure sheaf and the property*

$$\forall x \in \mathcal{X}\colon\ \mathrm{Cl}(\mathcal{O}_{\mathcal{X},x}) = 0;$$

then $\mathrm{CaDiv}(\mathcal{X}) = \mathcal{D}_{\mathrm{coh}}(\mathcal{X})$.

2. *Let $\mathcal{O}$ be a local ring with maximal ideal $\mathcal{M}$ and $\mathcal{X}$ be an integral $\mathcal{O}$-scheme with coherent structure sheaf and closed structure morphism. If in addition $\mathcal{X}$ satisfies*

$$\forall x \in \mathcal{X} \times_{\mathcal{O}} \mathcal{O}/\mathcal{M}\colon \mathrm{Cl}(\mathcal{O}_{\mathcal{X},x}) = 0,$$

then $\mathrm{CaDiv}(\mathcal{X}) = \mathcal{D}_{\mathrm{coh}}(\mathcal{X})$.

In particular, this holds for $\mathcal{O}_v$-varieties with closed structure morphism.

Proof: The proof of point 1 is standard and will be omitted.

Let ϕ be the structure morphism of $\mathcal{X}|\mathcal{O}$. The set $\phi(\overline{\{x\}}) \subseteq \mathrm{Spec}(\mathcal{O})$, $x \in \mathcal{X}$, is closed, hence contains the maximal ideal $\mathcal{M}$. This implies $\overline{\{x\}} \cap (\mathcal{X} \times_{\mathcal{O}} \mathcal{O}/\mathcal{M}) \neq \emptyset$; it follows that $\mathcal{O}_{\mathcal{X},x}$ is a localization of a local ring $\mathcal{O}_{\mathcal{X},y}$, $y \in \mathcal{X} \times_{\mathcal{O}} \mathcal{O}/\mathcal{M}$. By Lemma 2.13, lifting of ideals in the flat extension $\mathcal{O}_{\mathcal{X},x}|\mathcal{O}_{\mathcal{X},y}$ gives a surjective homomorphism $\mathcal{D}^v_{\mathrm{fin}}(\mathcal{O}_{\mathcal{X},y}) \to \mathcal{D}^v_{\mathrm{fin}}(\mathcal{O}_{\mathcal{X},x})$, which maps principal ideals to principal ideals. It therefore gives rise to a surjective homomorphism $\mathrm{Cl}(\mathcal{O}_{\mathcal{X},y}) \to \mathrm{Cl}(\mathcal{O}_{\mathcal{X},x})$. Since $\mathrm{Cl}(\mathcal{O}_{\mathcal{X},y}) = 0$ by assumption this proves the assertion by point 1. ∎

For the rest of the section let $\mathcal{X}$ again be a normal variety over the valuation ring $\mathcal{O}_v$.

For any point $x \in \mathcal{X}$, the local ring $\mathcal{O}_{\mathcal{X},x}$ by Corollary 2.5 and Theorem 1.11 is a localization of a PvM-ring and therefore a PvM-ring itself ([Zaf], Cor. 11). Consequently $\mathcal{D}^v_{\mathrm{fin}}(\mathcal{O}_{\mathcal{X},x})$ resp. $\mathrm{Div}(\mathcal{O}_{\mathcal{X},x})$ are groups. In the study of their structure it suffices to investigate the case $x \in Xv = \mathcal{X} \times_{\mathcal{O}_v} Kv$: If $x \in X = \mathcal{X} \times_{\mathcal{O}_v} K$ the local ring $\mathcal{O}_{\mathcal{X},x}$ is a localization of a finitely generated normal K-algebra, hence noetherian. Furthermore, any point $x \in \mathcal{X} \times_{\mathcal{O}_v} \kappa(\mathcal{P})$, $\mathcal{P} \in \mathrm{Spec}(\mathcal{O}_v) \setminus \{0, \mathcal{M}_v\}$, lies on the closed fibre of the normal $(\mathcal{O}_v)_{\mathcal{P}}$-variety $\mathcal{X} \times_{\mathcal{O}_v} (\mathcal{O}_v)_{\mathcal{P}}$.

To state the subsequent results in a geometrically intuitive way we introduce the following notations: By Theorem 2.6 1.+3., the irreducible components of the closed fibre Xv of $\mathcal{X}$ are in bijection with the valuations in the set V; for any $\mathrm{v} \in \mathrm{V}$ the irreducible component corresponding to v via this bijection is denoted by $X\,\mathrm{v}$.

Define rings

$$\mathcal{O}_{\mathrm{V}_x} := \bigcap_{\mathrm{v} \in \mathrm{V}\,:\, x \in X\,\mathrm{v}} \mathcal{O}_{\mathrm{v}}, \quad R_x := \bigcap_{P \in \mathcal{P}(\mathcal{X})\colon x \in \mathbf{P}} \mathcal{O}_{v_P}.$$

The ring $\mathcal{O}_{\mathrm{V}_x}$ is — for the same reasons as the ring $\mathcal{O}_{\mathrm{V}}$ defined earlier in this paper — a semilocal Prüfer ring.

THEOREM 2.15: *Let $\mathcal{X}$ be a normal $\mathcal{O}_v$-variety and $x \in Xv$. Then:*

1. $R_x = (\mathcal{O}_v \setminus 0)^{-1}\mathcal{O}_{\mathcal{X},x}$*; in particular* R_x *is noetherian.*
2. $\mathcal{O}_{\mathcal{X},x} = R_x \cap \mathcal{O}_{\mathrm{V}_x}$*; in particular* $\mathcal{O}_{\mathcal{X},x}$ *is a ring of Krull type.*
3. $\mathrm{Div}(\mathcal{O}_{\mathcal{X},x}) = \mathrm{Div}(R_x) \oplus \mathrm{Div}(\mathcal{O}_{\mathrm{V}_x})$.

Proof: 1. The ring $(\mathcal{O}_v \setminus 0)^{-1}\mathcal{O}_{\mathcal{X},x}$ is noetherian since it is a localization of a finitely generated K-algebra. It is normal by assumption and therefore a Krull ring; the defining set of discrete valuation rings is given by the localizations at the prime ideals of height one, hence this set is equal to

$$\{(\mathcal{O}_{\mathcal{X},x})_p \mid p \in \mathrm{Spec}(\mathcal{O}_{\mathcal{X},x})\colon \mathrm{ht}(p) = 1, p \cap \mathcal{O}_v = 0\} = \{\mathcal{O}_{v_P} \mid P \in \mathcal{P}(X)\colon x \in \mathbf{P}\}.$$

The latter set is precisely the defining set of discrete valuations of the Krull ring R_x. The assertion now follows, since Krull rings having the same set of defining valuation rings are equal.

2. According to (1.5 1.+3.) the assertion will be proved once one can show

$$\mathrm{MaxVal}(\mathcal{O}_{\mathcal{X},x}) = \{\mathcal{M}_{v_P} \cap \mathcal{O}_{\mathcal{X},x} \mid x \in \mathbf{P}\} \cup \{\mathcal{M}_{\mathrm{v}} \cap \mathcal{O}_{\mathcal{X},x} \mid x \in X\,\mathrm{v}\}.$$

The inclusion $\supseteq$ follows from Proposition 2.9. Since $\mathcal{O}_{\mathcal{X},x}$ is a localization of a finitely generated $\mathcal{O}_v$-algebra, Lemma 2.8 shows the reverse inclusion.

Finally, $\mathcal{O}_{\mathcal{X},x}$ is a ring of Krull type since R_x is of Krull type (even a Krull ring) and the semilocal Prüfer ring $\mathcal{O}_{\mathrm{V}_x}$ is of Krull type too.

3. According to point 2 the div-homomorphism associated to the PvM-ring $\mathcal{O}_{\mathcal{X},x}$ gives an embedding

$$\mathrm{div}_x : \mathcal{D}_{\mathrm{coh}}(\mathcal{O}_{\mathcal{X},x}) \to \mathrm{Div}(R_x) \oplus \prod_{\mathrm{v} \in \mathrm{V}\,:\, x \in X\,\mathrm{v}} \mathrm{v}\,F.$$

Any $D \in \mathrm{Div}(R_x)$ can be understood as an element of $\mathrm{Div}(X)$, therefore by Theorem 2.10 there exists $\mathcal{J} \in \mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ such that $\mathrm{div}(\mathcal{J}) = D$, where div denotes the div-homomorphism associated to $\mathcal{X}$. By Lemma 1.7 the stalk $\mathcal{J}_x$ is a divisorial $\mathcal{O}_{\mathcal{X},x}$-ideal and by definition of div_x one has $\mathrm{div}_x(\mathcal{J}_x) = D$, which shows that $\mathrm{Div}(R_x)$ is a direct summand of $\mathrm{Div}(\mathcal{O}_{\mathcal{X},x})$: $\mathrm{Div}(\mathcal{O}_{\mathcal{X},x}) = \mathrm{Div}(R_x) \oplus \Gamma$. Using the same reasoning as in the proof of Proposition 2.12 one obtains $\Gamma = \mathrm{Div}(\mathcal{O}_{\mathrm{V}_x})$. ∎

We close this subsection with the announced criterion for a $\mathcal{O}_v$-variety to satisfy $\mathrm{CaDiv}(\mathcal{X}) = \mathcal{D}_{\mathrm{coh}}(\mathcal{X})$:

THEOREM 2.16: *Let $\mathcal{X}$ be a normal $\mathcal{O}_v$-variety. Then:*

1. *For any point $x \in Xv$, one has*

$$\mathrm{Cl}(\mathcal{O}_{\mathcal{X},x}) = 0 \Rightarrow R_x \text{ is factorial.}$$

2. *If the structure morphism of $\mathcal{X}$ is closed and* $\mathrm{Gen}(Xv)$ *possesses an affine open neighborhood, one has*

$$t(\mathrm{Div}(X)) \subseteq \mathrm{CaDiv}(\mathcal{X}) \Leftrightarrow \mathrm{CaDiv}(\mathcal{X}) = \mathcal{D}_{\mathrm{coh}}(\mathcal{X}).$$

Proof: 1. R_x satisfies $R_x = (\mathcal{O}_v \setminus 0)^{-1}\mathcal{O}_{\mathcal{X},x}$; applying Lemma 2.13 gives an epimorphism $\ell\colon \mathcal{D}^v_{\mathrm{fin}}(\mathcal{O}_{\mathcal{X},x}) \to \mathcal{D}^v_{\mathrm{fin}}(R_x)$. Since ℓ maps principal fractionary ideals into such, it induces an epimorphism $\mathrm{Cl}(\mathcal{O}_{\mathcal{X},x}) \to \mathrm{Cl}(R_x)$. This shows that $\mathrm{Cl}(\mathcal{O}_{\mathcal{X},x}) = 0$ implies $\mathrm{Cl}(R_x) = 0$. Since a Krull ring with trivial class group is factorial ([Fos], Prop. 6.1) the assertion is verified.

2. Assume the invertibility of all sheaves in the set $t(\mathrm{Div}(X))$. By Corollary 2.11 and Proposition 2.12 every sheaf $\mathcal{J} \in \mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ has the form $\mathcal{J} = (t(D)\cdot\mathcal{J}_1)^\frown$, where $D \in \mathrm{Div}(X)$ and $\mathrm{div}(\mathcal{J}_1) \in \mathrm{Ver}(\mathcal{X})$ hold. By Proposition 2.12, 2. and since $\mathcal{O}_V$ is a semilocal Prüfer ring, there exists an $a \in \mathcal{O}_V$ such that $\mathrm{div}(a\mathcal{O}_{\mathcal{X}}) = E + \mathrm{div}(\mathcal{J}_1)$, $E \in \mathrm{Div}(X)$. The sheaf $t(E)$ is by assumption invertible, hence one obtains $\mathcal{J} = (t(D)\cdot a\mathcal{O}_{\mathcal{X}}\cdot t(E)^{-1})^\frown = t(D)\cdot a\mathcal{O}_{\mathcal{X}}\cdot t(E)^{-1} \in \mathrm{CaDiv}(\mathcal{X})$.

The other implication is obvious. ∎

Theorem 2.16 has an interesting connection to results obtained by P. Roquette and E. Kani in other contexts. This connection demonstrates applications of the results in the present paper, and it also shows the direction of further investigations: P. Roquette has proved in [Roq2] that for a proper normal curve $\mathcal{X}$ over a valuation ring $\mathcal{O}_v$ with algebraically closed field of fractions and normal closed fibre Xv, the ring R_x is a principal ideal domain (Theorem 4.3). This is equivalent to saying that R_x is factorial: The function field $F|K$ of $\mathcal{X}$ is of transcendence degree 1 and R_x is normal and contains K, thus it is a Dedekind ring. He formulates and proves this result in the language of valued function fields, but with the help of [GMP], Theorem 2.1 it can be translated into the form stated above.

E. Kani has proved the inclusion $t(\mathrm{Div}(X)) \subseteq \mathrm{CaDiv}(\mathcal{X})$ in point 2 for $\mathcal{O}_v$-curves $\mathcal{X}$ of the form $C \times_L \mathcal{O}_v$, $\mathcal{O}_v$ an arbitrary valuation ring, $L \subseteq \mathcal{O}_v$ a field and C a smooth L-curve ([Kan1], Hauptsatz 3.3 and [Kan2], Hauptsatz 3.6).

Both results can be deduced from Theorem 2.16 in the same way one would do this in the case of a noetherian ring $\mathcal{O}_v$: The noetherian notion of regularity of a scheme can be generalized to non-noetherian schemes $\mathcal{X}$ with coherent structure sheaf in such a way that a version of the theorem of Auslander–Buchsbaum holds: The local class groups $\mathrm{Cl}(\mathcal{O}_{\mathcal{X},x})$, $x \in \mathcal{X}$, of such a scheme are all equal to 0.

It can be shown that a smooth $\mathcal{O}_v$-scheme $\mathcal{X}$, where $\mathcal{O}_v$ is an arbitrary valuation ring, is regular in the generalized sense; thus its local class groups are 0. Since the $\mathcal{O}_v$-curves considered by Roquette and Kani are smooth, applying Theorem 2.16 gives the results cited above.

The author will return to this subject in a subsequent paper — [Kn2].

References

[Bou] N. Bourbaki, *Commutative Algebra,* Addison-Wesley, Reading, 1972.

[Bre-Hei] J. W. Brewer and W. J. Heinzer, *Associated primes of principal ideals,* Duke Mathematical Journal **41** (1974), 2–7.

[EGA1] A. Grothendieck and J. A. Dieudonne, *Elements de Geometrie Algebrique I,* Springer, Berlin–Heidelberg–New York, 1971.

[End] O. Endler, *Valuation Theory,* Springer, Berlin–Heidelberg–New York, 1972.

[Fos] R. M. Fossum, *The Divisor Class Groups of a krull Domain,* Springer, Berlin–Heidelberg–New York, 1973.

[G] B. Green, *On curves over valuation rings and morphisms to* $\mathbb{P}^1$, Journal of Number Theory **59** (1996), 262–290.

[Gil] R. Gilmer, *Multiplicative ideal theory,* Queen's Papers on Pure and Applied Mathematics 12, New York, 1972.

[Gla] S. Glaz, *Commutative coherent rings,* Lecture Notes in Mathematics **1371**, Springer, Berlin–Heidelberg–New York, 1989.

[GMP] B. W. Green, M. Matignon and F. Pop, *On valued function fields III, Reductions of algebraic curves,* Journal für die Reine und Angewandte Mathematik **432** (1992), 117–133.

[Gri1] M. Griffin, *Some results on v-multiplication rings,* Canadian Journal of Mathematics **19** (1967), 710–722.

[Gri2] M. Griffin, *Rings of Krull type,* Journal für die Reine und Angewandte Mathematik **229** (1968), 1–26.

[Ha] R. Hartshorne, *Algebraic Geometry,* Springer, Berlin–Heidelberg–New York, 1977.

[Jaf] P. Jaffard, *Les systèmes d'idèaux,* Travaux et recherches mathematiques, Dunod, Paris, 1960.

[Kan1] E. Kani, *Nonstandard Diophantische Geometrie,* Doctoral Thesis, University of Heidelberg, Heidelberg, 1978.

[Kan2] E. Kani, *Eine Verallgemeinerung des Satzes von Castelnuovo - Severi*, Journal für die Reine und Angewandte Mathematik **318** (1980), 178–220.

[Kap] I. Kaplansky, *Commutative Rings*, Allyn and Bacon, Boston, 1970.

[Kn1] H. Knaf, *Divisoren und Schnittprodukte auf Kurven über Bewertungsringen*, Doctoral Thesis, University of Heidelberg, Heidelberg, 1997.

[Kn2] H. Knaf, *Regularity on varieties over Prüfer domains*, Manuscript, Heidelberg, 1998.

[Mat] H. Matsumura, *Commutative Algebra*, Benjamin, London–Amsterdam–Sydney–Tokyo, 1980.

[Mer] J. Merker, *Idéaux faiblement associés*, Bulletin des Sciences Mathématiques **93** (1969), 15–21.

[Mot-Zaf] J. L. Mott and M. Zafrullah, *On Prüfer v-multiplication domains* Manuscripta Mathematica **35** (1981), 1–26.

[Nag] M. Nagata, *Finitely generated rings over a valuation ring*, Journal of Mathematics of Kyoto University **5** (1965), 163–169.

[Pes] C. Peskine, *Une généralisation du Main Theorem de Zariski*, Bulletin des Sciences Mathématiques **90** (1966), 119–127.

[Roq1] P. Roquette, *Zur Theorie der Konstantenreduktion algebraischer Mannigfaltigkeiten*, Journal für die Reine und Angewandte Mathematik **200** (1958), 1–44.

[Roq2] P. Roquette, *Reciprocity in valued function fields*, Journal für die Reine und Angewandte Mathematik **375/376** (1987), 239–258.

[Sab] G. Sabbagh, *Coherence of polynomial rings and bounds in polynomial ideals*, Journal of Algebra **31** (1974), 499–507.

[Vas1] W. V. Vasconcelos, *The Rings of Dimension Two*, Marcel Dekker, New York–Basel, 1976.

[Vas2] W. V. Vasconcelos, *Divisor Theory in Module Categories*, Mathematics Studies 14, North-Holland, Amsterdam - Oxford, 1974.

[Zaf] M. Zafrullah, *On finite conductor domains*, Manuscripta Mathematica **24** (1978), 191–204.